SOLUTION OF TRIANGLES & INVERSE TRIGONOMETRIC FUNCTIONS

Relations between the sides and angles of a triangle

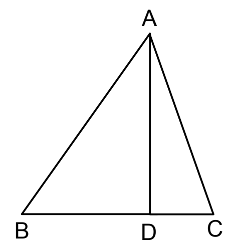
(i) **SINE RULE:**

In any Δ ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

i.e. the sines of the angles are proportional to the lengths of the opposite sides.

Proof: Let AD be perpendicular from A on BC.

In Δ ABD, we have $\sin B = \frac{AD}{AB} \Rightarrow AD = c \sin B$



In Δ ACD, we have

$\sin C = \frac{AD}{AC} \Rightarrow AD = b \sin C$

$\therefore$ AD = b sin C = c Sin B $\Rightarrow \frac{\sin B}{b} = \frac{\sin C}{c}$ …..(i)

Similarly, $\frac{\sin A}{a} = \frac{\sin B}{b}$ …….(ii)

From (i) and (ii), we have $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

NOTE: (1) The above rule can also be written as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(2) The sine rule is generally used to express sides of the triangle in terms of sines of angles and vice-versa as discussed below -

Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

Then, a = k sin A, b = k sin B, c = k sin C

Or sin A = $\frac{1}{k}$ a, sin B = $\frac{1}{k}$ b, sin C = $\frac{1}{k}$ c.

(ii) **COSINE FORMULAE**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

or

$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}$

$\mathbf{b^2 = c^2 + a^2 - 2ca \cos B}$

$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C}$

Proof: In Δ ACD we have

$AC^2 = AD^2 + CD^2 = AD^2 + (BC - BD)^2$

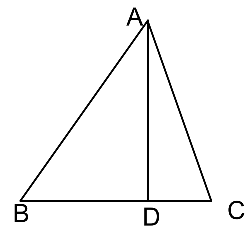
$\Rightarrow AC^2 = AD^2 + BC^2 + BD^2 - 2BC.BD$

$\Rightarrow AC^2 = BC^2 + (AD^2 + BD^2) - 2BC.\ AB \cos B \quad \left[\because \cos B = \frac{BD}{AB}\right]$

$\Rightarrow AC^2 = BC^2 + AB^2 - 2BC.\ AB \cos B$

$\Rightarrow b^2 = a^2 + c^2 - 2ac \cos (B)$

$\Rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac}$



Similarly, we can prove the other two formulae.

(iii) **PROJECTION FORMULAE**

a = b cos c + c cos B

b = c cos A + a cos C

c = a cos B + b cos A

Proof: In triangles ABD and ACD (above fig) we have BD = AB cos B and CD = AC cos C.

$\therefore$ a = BC = BD + CD $\Rightarrow$ a = AB cos B + AC cos C $\Rightarrow$ a = c cos B + b cos C

(iv) **NAPIER'S ANALOGY OR TANGENT FORMULA**

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

(V) **SEMI-SUM FORMULAE**

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} \qquad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} \qquad \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} \qquad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \frac{2\Delta}{bc}$$

$$\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \frac{2\Delta}{ac}$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \frac{2\Delta}{ab}$$

Note:

$$\Delta = \tfrac{1}{2}ab\sin C$$
$$= \tfrac{1}{2}bc\sin A$$
$$= \tfrac{1}{2}ac\sin B$$
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

(vi) **CIRCUM-CIRCLE and IN-CIRCLE of a triangle**

CIRCUM-CIRCLE:

Definition: The circle which passes through the three vertices of a triangle is called *circumcircle* or *circumscribing* circle of the triangle. Its centre is called the *circumcentre* and the radius is called the *circumradius* & is denoted by R.

The radius of the circumcircle of a triangle ABC i.e. the **circumradius(R)** is given by -

(i) $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

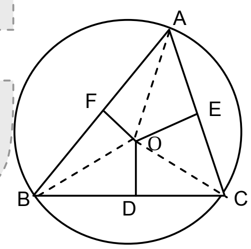
(ii) $R = \dfrac{abc}{4\Delta}$

Proof: (i) In the Fig the perpendicular bisectors of the sides BC, CA and AB intersect at O.

Therefore O is the circumcentre such that OA = OB = OC = R. We have $\angle$ BOC = 2$\angle$BAC = 2(A)

Therefore, $\angle$ BOD =$\angle$COD = (a)

In Δ OBD, sin A = $\dfrac{BD}{OB} = \dfrac{a/2}{R} \Rightarrow R = \dfrac{a}{2\sin A}$



Similarly, $R = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

Hence, $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

(ii) $\Delta = \dfrac{1}{2}bc\ \sin A \Rightarrow \sin A = \dfrac{2\Delta}{bc}\ R = \dfrac{a}{2\sin A}\ \Rightarrow R = \dfrac{a}{2(2\Delta/bc)} = \dfrac{abc}{4\Delta}.$

IN-CIRCLE:

Definition: A circle which touches all the three sides of a triangle internally is called the *incircle* of the triangle. Its centre is called the *incentre* and the radius is called *inradius* & is denoted by r.

The radius of the incircle of a triangle ABC i.e. the **inradius(r)** is given by -

(i) $\mathbf{r} = \dfrac{\Delta}{s}$

(ii) $\mathbf{r = (s - a)\ \tan}\ \dfrac{A}{2}$

$\mathbf{= (s - b)\ \tan}\ \dfrac{B}{2}$

$\mathbf{= (s - c)\ \tan}\ \dfrac{C}{2}$

Area of Cyclic Quadrilateral Let ABCD be a cyclic quadrilateral whose sides AB, BC, CD and DA are respectively a, b, c and (d)

The area of quadrilateral = $\sqrt{[(s-a)(s-b)(s-c)(s-d)}$

Note: (i) Remember **cos B =** $\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

(ii) If ABCD is a cyclic quadrilateral, then **AC. BD = A(B) CD + BC. AD** (Ptolemy's Theorem)

INVERSE TRIGONOMETRIC (OR CIRCULAR) FUNCTIONS

INVERSE FUNCTION	DOMAIN	RANGE(value of y) (Principle values of the functions)
$y = \sin^{-1} x$	$\vert x \vert \le 1$ i.e. $[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$\vert x \vert \le 1$ i.e. $[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	$x \in R$ i.e. $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	$x \in R$ i.e. $(-\infty, \infty)$	$(0, \pi)$
$y = \sec^{-1} x$	$\vert x \vert \ge 1$ i.e. $(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ i.e. $[0, \pi] - \{\frac{\pi}{2}\}$
$y = \text{cosec}^{-1} x$	$\vert x \vert \ge 1$ i.e. $(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ i.e. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

NOTE: As we have seen that none of the trigonometric function is 1 – 1 (one-one). Thus, in order to define inverse trigonometric functions, we will have to restrict the domain of these functions so that in the restricted domain, the functions may be 1 – 1.

Ex.6 **Principal Value of $\sin^{-1}$ ½ and cosec^{-1} (– 1)**

Sol. (i) $y = \sin^{-1} \frac{1}{2}$

$\sin y = \frac{1}{2}$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ $y = \frac{\pi}{6}$, because $\sin \frac{\pi}{6} = \frac{\pi}{2}$

$\frac{1}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ Principal value of $\sin^{-1}$ ½ = $\pi/6$

(ii) $y = \text{cosec}^{-1}(-1)$ $\therefore$ cosec y = – 1 and $y \in [-\pi/2, \pi/2] - \{0\}$.

Now, cosec $\left(-\frac{\pi}{2}\right) = -\text{cosec}\frac{\pi}{2} = -1$

$\therefore$ Principal value of $\text{cosec}^{-1}(-1) = -\pi/2$.

IMPORTANT RESULTS

$\sin^{-1} x + \cos^{-1} x = \pi/2$, $|x| \le 1$

$\tan^{-1} x + \cot^{-1} x = \pi/2$, $x \in R$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$, $|x| \ge 1$

$\sin^{-1}(-x) = -\sin^{-1} x$, $|x| \le 1$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$, $|x| \le 1$

$\tan^{-1}(-x) = -\tan^{-1} x$, $x \in R$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, $|x| \ge 1$

$\sec^{-1}(-x) = \pi - \sec^{-1} x$, $|x| \ge 1$

$\cot^{-1}(-x) = -\cot^{-1} x$, $x \in R$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, $|x| \ge 1$

$\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$, 0 , $|x| \le 1$

$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, $|x| \ge 1$.

$\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$, $0 < |x| \le 1$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$, $x > 0$

$\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$, $x > 0$

$\sin(\cos^{-1} x) = \cos(\sin^{-1} x) = \sqrt{1-x^2}$, $|x| \le 1$

$\sec(\operatorname{cosec}^{-1} x) = \operatorname{cosec}(\sec^{-1} x) = \frac{|x|}{\sqrt{x^2-1}}$, $|x| > 1$

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy < 1$

$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $xy > -1$

$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$, $x < 1$

$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4} - \tan^{-1} x$, $x > -1$

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x$, $|x| \le 1$

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x$, $x \ge 0$

$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$, $|x| < 1$

$2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$, $|x| \le 1/\sqrt{2}$

$2\cos^{-1} x = \cos^{-1}(1 - 2x^2),\ x \in [0, 1]$
$3\sin^{-1} x = \sin^{-1}(3x - 4x^3),\ |x| \le ½$
$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x),\ ½ \le x \le 1$
$\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\sin^{-1} x - \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$
$\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-y^2}\sqrt{1-x^2})$
$\cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy + \sqrt{1-y^2}\sqrt{1-x^2})$

Ex. Prove that

(i) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

(ii) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

Sol. (i) L.H.S. $= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} \quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{1}\left(\frac{3+2}{6-1}\right) = \tan^{-1} 1 = \frac{\pi}{4} = \text{R.H.S.}$$

(ii) L.H.S. $= \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right) + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{4}{5}\cdot\frac{12}{13} + \frac{5}{13}\cdot\frac{3}{5}\right) + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65} = \sin^{-1}\frac{63}{65} + \cos^{-1}\sqrt{1-\left(\frac{16}{65}\right)^2}$$

$$= \sin^{-1}\frac{63}{65} + \cos^{-1}\frac{63}{65} = \frac{\pi}{2}. \quad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

Ex. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$. Show that $x^2 + y^2 + z^2 + 2xyz = 1$.

Sol. We have $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$.

$\therefore \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}] = \pi - \cos^{-1} z = \cos^{-1}(-z)$

$\therefore xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\therefore (xy + z)^2 = (1 - x^2)(1 - y^2)$

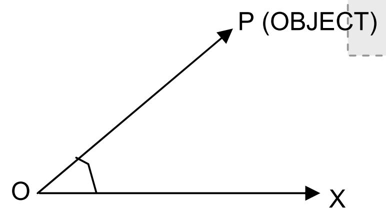
$\therefore x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\therefore x^2 + y^2 + z^2 + 2xyz = 1.$

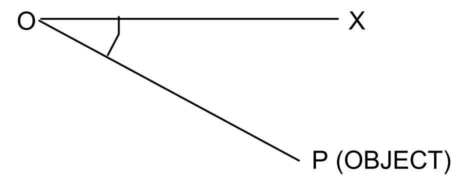
HEIGHTS AND DISTANCES

Sometimes, it becomes very difficult to actually measure the distances between two points or heights of certain objects. The methods of trigonometry help us in finding these heights and distances in a very simple and easier manner.

Let *P* be the position of an object and OX be the horizontal axis.



(i) ANGLE OF ELEVATION

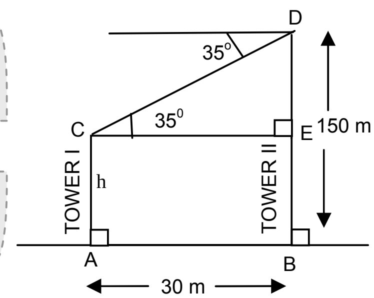


(ii) ANGLE OF DEPRESSION

If *P* is above *OX*, then angle *XOP* is called the *angle of elevation* of *P* as observed from *O*.
If *P* is below *OX*, the angle *XOP* is called the *angle of depression* of *P* as observed from *O*.

Ex. The horizontal distance between two towers is 30 m and the angular depression of the top of the first as seen from the top of the second, which is 150 m high is 35^0. Find the height of the first tower.

Sol. Let h m be the height of the first tower.



In Δ CED,

$$\frac{DE}{CE} = \tan 35^0$$

or $$\frac{BD - AC}{AB} = \tan 35^0$$

or $$\frac{150 - h}{30} = \tan 35^0$$

$\therefore$ h = 150 – 30 tan 35^0

= 150 – 30 (.7002) = 128.994 m.