PROBABILITY

There are two approaches to probability viz. (i) classical approach and (ii) axiomatic approach. The classical approach was developed by a French mathematician (B) Pascal whereas the axiomatic approach was developed by a Russian mathematician (A) Kolmogorov in 1933. In both the approaches we frequently use the term 'experiment', which means an operation which can produce some well-defined outcome(s). There are two types of experiments.

(i) Deterministic experiment **(ii) Probabilistic experiment or Random experiment**

(i) <u>Deterministic Experiment:</u> Those experiments which when repeated under identical conditions produce the same result or outcome, are known as **deterministic experiment.**

***For example* -** When experiment in science or engineering are repeated under identical conditions, we get almost the same result every time.

(ii) <u>Probabilistic or Random Experiment:</u> If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as a **probabilistic experiment or a random experiment.**

For example: In tossing of a coin one is not sure if a head or a tail will be obtained, so it is a random experiment. Similarly rolling an unbiased die and drawing a card from a well shuffled pack of cards are examples of random experiments.

SOME IMPORTANT DEFINITIONS

<u>Trial and Elementary Events:</u> Let a random experiment be repeated under identical conditions.

Then the experiment is called a **trial** and the possible outcomes of the experiment are known as **elementary events or cases (or indecomposable events)**.

***For example*:**

i Tossing of a coin is a trail and getting head or tail is an elementary event.

ii Throwing of a die is a trail and getting 3 on its upper face is an elementary event.

iii Drawing a card from a pack of well-shuffled cards is a trail and getting a king of diamond is an elementary event.

iv Drawing two balls from a bag containing 4 white and 6 red balls is a trail and getting one red and one white ball is an elementary event.

<u>Compound Events:</u> Events obtained by combining together two or more elementary events are known as the **compound events or decomposable events.**

Compound event is said to occur if one of elementary events associated with it occurs.

***For example*:**

- In a throw of a die the event: getting a multiple of 2, is a compound event because this event occurs if any one of the elementary events 2, 4 or 6 occurs. Similarly in a single throw of a pair of dice the event: getting a doublet is a compound event because this event occurs if any one of the elementary events (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) occurs.

- If a card is drawn from a well-shuffled pack of 52 playing cards. Then the event: getting a king, is a compound event because this event happens if any one of the four kings appears in the draw.

Exhaustive Number of Cases: The total number of possible outcomes of a random experiment in a trial is known as the **exhaustive number of cases**.

For example:

i. In throwing of a die the exhaustive number of cases is 6, since any one of the six faces marked with 1, 2, 3, 4, 5, 6 may come uppermost.

ii. If a pair of dice is thrown, then the exhaustive number of cases is 36, since any one of the six numbers 1, 2, 3,….., 6 on one die can be associated with any one of the six numbers on the other die.

iii. In drawing 4 cards from a well–shuffled pack of 52 cards, the exhaustive number of cases is ${}^{52}C_4$, since 4 cards can be drawn out of 52 cards in ${}^{52}C_4$ ways.

iv. In drawing two balls from a bag containing 3 white and 5 black balls the exhaustive number of cases is ${}^{8}C_2$, since 2 balls can be drawn out of 8 balls in ${}^{8}C_2$ ways.

Note: **If n dice are thrown then total possible outcomes = 6^n**
If n coins are thrown then total possible outcomes = 2^n

Mutually Exclusive Events: Events are said to be **mutually exclusive or incompatible** if the occurrence of any one of them prevents the occurrence of all the others.
i.e., if no two or more of them can occur simultaneously in the same trail.
The events that are not mutually exclusive are known as **compatible events**.

Note:

- *Elementary events* related to a random experiment are always mutually exclusive.
- *Compound events* may or may not be mutually exclusive.

For example:

In a single throw of a die, if we define the following compound events -

- E: getting an even number, F: getting a multiple of 3.
  These two events are not mutually exclusive, because when 6 appears on die we say that both E and F occurs together.

If we consider the following events -

- E: getting an even number, F: getting an odd number.
  These two events are mutually exclusive because, if E occurs we say that the number obtained is even and so it cannot be odd, i.e., F does not occur.
- In rolling of two dice, the events of the face marked 5 appearing on one die and face marked 5 appearing on the other are not mutually exclusive because both can happen together also.

Let S be sample space associated with a random experiment and let A_1 and A_2 be two events. Then A_1 and A_2 are mutually exclusive events if $\mathbf{A_1 \cap A_2 = \phi}$.

Mutually Exclusive and Exhaustive System of Events: Let S be the sample space associated with a random experiment.
Let $A_1, A_2, \ldots, A_n$ be subsets of S such that
(i) $A_1 \cap A_2 = \phi$ for $i \neq j$. and (ii) $A_1 \cup A_2 \cup \ldots \cup A_n = S$.
Then the collection of events $A_1, A_2, \ldots A_n$ said to form a mutually exclusive and exhaustive system of events.

If $E_1, E_2, \ldots\ldots E_n$ are elementary events associated with a random experiment. Then
(i) $E_1 \cap E_2 = \phi$ for $i \neq j$. and (ii) $E_1 \cup E_2 \cup \ldots \cup E_n = S$.

So, the collection of elementary events associated with a random experiment always forms a system or mutually exclusive and exhaustive system of events.

Equally Likely Events: Events are equally likely if there is no reason for an event to occur in preference to any other event.
***For example*:**
If an unbiased die is rolled, then each outcome is equally likely to happen, i.e., all elementary events are equally likely. If however, the die is so formed that a face marked 4 is heavier than the other faces, then the number marked on the opposite face will occur more often than the other faces. So, in this case, the events are not equally likely to occur.

Favourable Number of cases: The number of cases favourable to an event in a trial is the number of elementary events such that if any one of them occurs, we say that the event happens.
In other words, the number of cases favourable to an event in a trail is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.
For example:

i. In tossing of a die, the number of cases favourable to the appearance of multiple of 3 are two viz. (3 and 6)
ii. In throwing of two dice, the number of cases favourable to getting 8 as the sum are five i.e. (2, 6), (6, 2), (4, 4) (3, 5), (5, 3).
iii. In drawing two cards from a pack of 52 cards, the number of cases favourable to drawing 2 aces are $^4C_2$.

Independent Events: Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.
For example, if two cards are drawn from a well-shuffled pack of 52 cards after the other with replacement, then getting an ace in the first draw is independent to getting a king in the second draw. But, if the first card drawn in the first draw is not replaced, then second draw is dependent on the first draw.
If two dice are thrown together, then getting an even number on first is independent to getting an odd number in the second draw.

Sample Space: The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.
If $E_1, E_2, \ldots\ldots, E_n$ are the possible outcomes of a random experiment, then $S = \{E_1, E_2, \ldots\ldots E_n\}$.
Each element of S is called a sample point.

Impossible and Certain Events: Let S be the sample space associated with a random experiment, then ϕ and S, being subsets of S, are events. The event ϕ is called an **impossible event** and the event S is known as a **certain event.**

DEFINITION OF PROBABILITY

Let S be the sample space associated with a random experiment, and let A be a subset of S representing an event. Then the probability of the event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of elementary events in A}}{\text{Number of elementary events in S}}$$

In other words, $P(A) = \dfrac{\text{Number of favourable events or cases}}{\text{Total Number of events(cases) i.e. Sample Space}}$

Also, $P(\overline{A}) = \dfrac{\text{Number of elementary events in } \overline{A}}{\text{Number of elemetary events in S}} = \dfrac{n(S)-n(A)}{n(S)} = 1 - \dfrac{n(A)}{n(S)} = 1 - P(A).$

Points to remember:

- If an event A is sure to occur then P (A) = 1
- If an event A is sure not to occur then P(A) = 0
- $0 \le P(A) \le 1$
- $P(A) + P(\overline{A}) = 1$
- The **odds in favour** of occurrence of the event A are defined by, $P(A) : P(\overline{A})$
- The **odds against** the occurrence of A are defined by $P(\overline{A}) : P(A)$

EXAMPLES

1. An unbiased dice is thrown. What is the probability of getting (i) an even number (ii) a multiple of 3 (iii) an even number or a multiple of 3 (iv) an even number & a multiple of 3?

Sol. In a single throw of an unbiased we can get any one of the outcomes 1, 2, 3, 4, 5, 6. So, exhaustive number of cases = 6.

(i) An even number is obtained if we obtain any one of 2, 4, 6 as an outcome. So, favourable number of cases = 3. Thus, required probability = 3/6 = ½.

(ii) A multiple of 3 is obtained if we obtain any one of 3, 6 as an outcome. So, favourable number of cases = 2. Thus, required probability = 2/6 = 1/3.

(iii) An even number or a multiple of 3 is obtained in any of the following outcomes 2, 3, 4, 6. So, favourable number of cases = 4. Thus, required probability = 4/6 = 2/3.

(iv) An even number and a multiple of 3 is obtained if we get 6 as an outcome. So, favourable number of cases = 1. Thus, required probability = 1/6

2. Three unbiased coins are tossed. What is the probability of getting (i) all heads (ii) two heads (iii) one head (iv) at least one head (v) at least two heads?

Sol. If three coins are tossed together we can obtain any one of the following as an outcome.

HHH, HHT, HTH, THH, TTH, THT, HTT, TTT

So, exhaustive number of cases = 8.

(i) All heads can be obtained in only one way i.e., HHH.
So, favourable number of cases = 1. Thus, required probability = 1/8.

(ii) Two heads can be obtained in any one of the following ways; HHT, THH, HTH. So, favourable number of cases = 3. Thus, required probability = 3/8.

(iii) One head can be obtained in any one of the following ways: HTT, THT, TTH
So, favourable number of cases = 3. Thus, required probability = 3/8.

(iv) At least one head can be obtained in any one of the following ways:

HTT, THT, TTH, HHT, THH, HTH, HHH

So, favourable number of cases = 7. Thus, required probability = 7/8.

(v) At least two heads can be obtained in any one of the following ways:

HHH, HTH, THH, HHT

So, favourable number of cases = 4. Thus, required probability = 4/8 = ½

3. In a single throw of two dice what is the probability of getting
(i) 8 as the sum (ii) a doublet (iii) a total of 9 or 11 ?

Sol. In a single throw of a pair of dice, we can obtain any one of the following 36 cases as outcomes:

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (4, 4), (5, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

So, exhaustive number of cases = 36.

(i) The sum 8 can be obtained in any one of the following ways:

(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)

So, favourable number of cases = 5. Thus, required probability = 5/36

(ii) A doublet can be obtained in anyone of the following ways:

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

So, favourable number of cases = 6. Hence, required probability = 6/36 = 1/6.

(iii) A total of 9 or 11 can be obtained in any one of following ways:

(6, 3), (3, 6), (4, 5), (5, 4), (6, 5), (5, 6)

So, favourable number of cases = 6. Hence, required probability = 6/36 = 1/6

4. A coin is tossed successively three times. Find the probability of getting exactly one head or two heads?

Sol. Let S be the sample space associated with the given experiment. Then

S = {HHH, HHT, HTH, THH, HTT, TTH, THT, TTT}

Let A be the event of getting exactly one head or two heads. Then

A = {HHT, THH, HTH, TTH, HTT, THT}

Clearly n(A) = 6 and n(S) = 8. Therefore P(A) = $\frac{n(A)}{n(S)} = \frac{6}{8} = \frac{3}{4}$.

5. A die is thrown. Find the probability of getting (i) a prime number (ii) a multiple of 3 ?

Sol. Let S be the sample space associated with the experiment of throwing a die. Then S = {1, 2, 3, 4, 5, 6}.

(i) Let A be the event of selecting a prime number. Then A = {2, 3, 5}.

Therefore P(A) = $\frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

(ii) If A denotes the event "getting a multiple of 3". Then A = {3, 6}.

Therefore P(A) = $\frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$.

6. One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is:

(i) an ace, **(ii) red,**

(iii) either red or king, **(iv) red and a king ?**

Sol. Out of 52 cards, one card can be drawn in ${}^{52}C_1$ ways. Therefore exhaustive number of cases = ${}^{52}C_1 = 52$.

(i) There are four aces in a pack of 52 cards, out of which one can be drawn in ${}^{4}C_1$ ways. Therefore favourable number of cases = ${}^{4}C_1 = 4$. So, required probability = 4/52 = 1/13.

(ii) There are 26 red cards, out of which one red card can be drawn in ${}^{26}C_1$ ways. Therefore, favourable number of cases = ${}^{26}C_1 = 26$. So, required probability = 26/52 = ½.

(iii) There are 26 red cards including 2 red kings and there are 2 more kings. Therefore, there are 28 cards, one can be drawn in ${}^{28}C_1$ ways. Therefore favourable number of cases = ${}^{28}C_1 = 28$. So, required probability = 28/52 = 7/13.

(iv) There are 2 cards which are red and king. Therefore favourable number of cases = ${}^{2}C_1 = 2$. So, required probability = 2/52 = 1/26.

7. A bag contains tickets numbered 1 to 30. Three tickets are drawn at random from the bag. What is the probability that the maximum number of the selected tickets exceeds 25?

Sol. It is given that the maximum number on the selected tickets exceeds 25. This means that at least one of the selected tickets should bear a number that exceeds 25. Note that the negation of 'at least one' is none and in this case it will be easier for us to find the probability that one of selected tickets bear number exceeding 25.

Let A be the event that none of the selected tickets bear number exceeding 25. Then, required probability = P $(\overline{A})$ = 1 – P(A). Now we calculate P(A). The total number of ways of drawing three tickets out of 30 is ${}^{30}C_3$. Therefore Exhaustive number of cases = ${}^{30}C_3$.

Since none of the selected tickets bear number exceeding 25, therefore three tickets are drawn from tickets bearing number 1 to 25. This can be done in ${}^{25}C_3$ ways.

Therefore Favourable number of cases = ${}^{25}C_3$. So, P (A) = $\frac{{}^{25}C_3}{{}^{30}C_3} = \frac{115}{203}$.

Hence the required probability = P $(\overline{A})$ = $1 - \frac{115}{203} = \frac{88}{203}$

8. A card is drawn from an ordinary pack of 52 cards and a gambler bets that it is a spade or an ace. What are the odds against his winning this bet?

Sol. Let A be the event of getting a spade or an ace of spade and three aces other than an ace of spade.

Therefore, favourable number case = ${}^{16}C_1 = 16$. So, P(A) = $\frac{16}{52} = \frac{4}{13}$.

Hence, odds against A are P $(\overline{A})$: P(A) or $\frac{9}{13} : \frac{4}{13}$ or 9 : 4.

ALGEBRA OF EVENTS

We shall now discuss some standard methods of constructing new events in terms of some given events associated with a random experiment.

Consider the random experiment of throwing an unbiased die. Let A and B be two events associated with it such that A = getting even number, B = getting a multiple of 3.

A and B are described by the set A = {2, 4, 6}, B = {3, 6}.

"A or B" - Event "A or B" occurs iff A or B or both occur, i.e. at least one of A, B occurs.

"A or B" = {2, 3, 4, 6} = $A \cup (B)$

"A and B" - Event "A and B" occurs iff both A and B occurs simultaneously. "A and B" occur when the outcome belongs to both A and (B)

"A and B" = {6} = $A \cap (B)$

"not A" - Event "not A" occurs when and only when A does not occur.

For instance, in a single throw of a die if Event A: an odd number. Then A = {1, 3, 5}

"not A" = {2, 4, 6} = $\overline{A}$. Is also called complementary event of A or negation of (A)

"A implies B" - Sometimes the occurrence of one event implies the occurrence of other.

Note: if A implies B, then $A \cap B = (A)$

For example, in a single throw of a die if Event A: getting 2 or 4 and Event B: getting an even number. Then A = {2, 4) and B = (2, 4, 6}. Clearly the occurrence of A implies the occurrence of B, for if 2 or 4 occur, we say that the outcome is an even number.

Verbal Description of Event	Equivalent Set Theoretic Notation
Not A	$\overline{A}$ *or A'*
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \overline{B}$
Neither A nor B	$\overline{A} \cap \overline{B} = \overline{(A \cup B)}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$

ADDITION THEOREMS ON PROBABILITY

So far we have calculated the probability of occurrence of an event by using the definition only. But sometimes it is not convenient to find the number of cases favourable to the occurrence of an event due to which the computation of probability from the definition only is not possible. In such cases we calculate the probability of the event from known probabilities of other events. This is possible only when the given event is expressible as the union of two or more events.

For example, in the random experiment of drawing 2 cards from a well shuffled pack of 52 cards the event "getting both red cards or both kings" can be expressed as the union of two events viz. A: getting two red cards, B: getting two kings.

Theorem 1 (Addition Theorem for two events)

If A and B are two events associated with a random experiment, then

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Sol. Let S be the sample space associated with the given random experiment.

Suppose the random experiment results in n mutually exclusive ways.

Then S contains n elementary events.

Let m_1, m_2 and m be the number of elementary events favourable to A, B and $A \cap B$ respectively.

Then $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$ and $P(A \cap B) = \frac{m}{n}$.

The number of elementary events favourable to A only is $m_1 - m$.

Similarly the number of elementary events favourable to B only is $m_2 - m$.

Since m elementary events are favourable to both A and B or both, i.e. ,

$(A \cup B)$ is $m_1 - m + m_2 - m + m \Rightarrow m_1 + m_2 - m$.

So, $P(A \cup B) = \frac{m_1 + m_2 - m}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m}{n} \quad \Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Corollary: If A and B are mutually exclusive events, then

$P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

This is the addition theorem for mutually exclusive events.

Theorem 2 (Addition Theorem for three events)

If A, B, C are three events associated with a random experiment, then

$$\mathbf{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$$

Corollary: If A, B, C are mutually exclusive events, then

$P(A \cap B) = P(B \cap C) = P(A \cap C) P(A \cap B \cap C) = 0.$

$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$

This is the addition theorem for three mutually exclusive events.

Theorem 3 Let A and B be two events associated with a random experiment. Then

(i) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$ (ii) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Proof: (i) Since $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive events such that $(A \cap \overline{B}) \cup (\overline{A} \cap B) = B$

$\Rightarrow \quad P(A \cap B) + P(\overline{A} \cap B) = P(B) \quad \Rightarrow P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$P(\overline{A} \cap B)$ is known as the probability of occurrence of B only.

(ii) Since $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive events such that $(A \cap B) \cup (A \cap \overline{B}) = A$

$P(A \cap B) + P(A \cap \overline{B}) = P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$P(A \cap \overline{B})$ is known as the probability of occurrence of A only.

Theorem 4 – If $B \subset A$, then prove that

(i) $P(A \cap \overline{B})$ **= P (A) – P (B)** (ii) $\mathbf{P(B) \le P(A)}$

Proof: (i) Since $A \cap \overline{B}$ and B are mutually exclusive events such that their union is (A) Therefore

$P(A) = P(B) + P\ P(A \cap \overline{B}) \quad \Rightarrow P(A \cap \overline{B}) = P(A) - P(B).$

(ii) Since $P(A \cap \overline{B}) \ge 0$, therefore $P(A \cap \overline{B}) = P(A) - P(B) \ge 0 \quad \Rightarrow \quad P(A) \ge P(B).$

EXAMPLES

1. A and B are two mutually exclusive events of an experiment. If P ('not A') = 0.65, P (A $\cup$B) = 0.65 and P(B) = p, find the value of p.

Sol. By addition theorem for mutually exclusive events, we have

$P(A\cup B) = P(A) + P(B) = 1 - P('\text{not } A') + P(B)$ [as $P(A) = 1 - P(\overline{A})$]

$\Rightarrow 0.65 = 1 - 0.65 + p \quad \Rightarrow p = 0.30$

2. The probability of two events A and B are 0.25 and 0.50 respectively. The probability of their simultaneous occurrence is 0.1. Find the probability that neither A nor B occurs.

Sol. We have P(A) = 0.25, P(B) = 0.50 and P(A $\cap$B) = 0.14.

Required probability = $P\left(\overline{A}\cap\overline{B}\right) = P\left(\overline{A\cup B}\right)$ $\left[\because \left(\overline{A\cup B}\right) = \overline{A}\cap\overline{B}\right]$

$= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [0.25 + 0.50 - 0.14] = 0.39.$

3. Find the probability of getting an even number on the first dice or a total of 8 in a single throw of two dice.

Sol. Let S be the sample space associated with the experiment of throwing a pair of dice. Then n(S) = 36.

Let A and B be two events given by

A = getting an even number on first die, B = getting a total of 8.

Then A $\cap$ B = getting an even number on first die and total of 8.

Now, A = {(2, 1),…,(2, 6), (4, 1), (4, 2),…. (4, 6), (6, 1), (6, 2),….., (6, 6)}

B = {(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)} and A $\cap$ B = {(2, 6), (6, 2), (4, 4)}

$\therefore$ P(A) = 18/36, P(B) = 5/36 and P (A $\cap$ B) = 3/36.

Required probability = P(A $\cup$ B) = P(A) + P(B) – P (A $\cap$ B) = (18/36) + (5/36) – (3/36) = 20/36 = 5/9.

4. Four cards are drawn at a time from a pack of 52 playing cards. Find the probability of getting all the four cards of the same suit.

Sol. Since 4 cards can be drawn at a time from a pack of 52 cards in ${}^{52}C_4$ ways, therefore

Exhaustive number of cases = ${}^{52}C_4$.

Consider the following events

A = getting all spade cards, B = getting all club cards, C = getting all diamond cards, and

D = getting all heart cards. Then A, B, C and D are mutually exclusive events such that

$P(A) = ({}^{13}C_4 / {}^{52}C_4)$, $P(B) = ({}^{13}C_4 / {}^{52}C_4)$, $P(C) = ({}^{13}C_4 / {}^{52}C_4)$ and $P(D) = ({}^{13}C_4 / {}^{52}C_4)$

Now, required probability = $P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$ [by ad(d) theorem]

$= 4\,[{}^{13}C_4 / {}^{52}C_4] = 44/4165$

5. Two cards are drawn from a pack of 52 cards. What is the probability that either both are red or both are kings?

Sol. Out of 52 cards, two cards can be drawn in ${}^{52}C_2$ ways. So, exhaustive number of cases = ${}^{52}C_2$.

Consider the following events.

A = Two cards drawn are red cards; B = Two cards drawn are kings.

Required probability = P(A $\cup$ B) = P(A) + P(B) – P(A $\cup$ B) … (i) [by ad(d) theorem]

Now we shall find P(A), P(B) and (A $\cap$ B).

There are 26 red cards, out of which 2 red cards can be drawn in ${}^{26}C_2$ ways $\therefore$ $P(A) = ({}^{26}C_2 / {}^{52}C_2)$

Since there are 4 kings, out of which 2 kings can be drawn in $\therefore$ $P(A) = ({}^{4}C_2 / {}^{52}C_2)$

There are 2 cards which are both red and kings, therefore

P(A $\cap$ B) = Probability of getting 2 cards which are both red and kings.

From (i), Required probability = P(A) + P(B) – P(A $\cap$ B) = $({}^{26}C_2 / {}^{52}C_2) + ({}^{4}C_2 / {}^{52}C_2) - ({}^{2}C_2 / {}^{52}C_2)$

= (325/1326) + (1/221) – (1/1326) = 55/221

6. From a well shuffled deck of 52 cards, 4 cards are drawn at random. What is the probability that all the drawn cards are of the same colour?

Sol. Let A = 4 cards drawn are red, B = 4 cards drawn are black. Then A, B are mutually exclusive events. So required probability = $P(A \cup B) = P(A) + P(B) = 2\,({}^{26}C_4 / {}^{52}C_4) = 92/883$

CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then the probability of occurrence of A under the condition that B has already occurred and $P(B) \neq 0$ is called the **conditional probability** and it is denoted by P (A/B).
Thus, P (A/B) = Probability of occurrence of A given that B has already happened.
Similarly, P (B/A) = Probability of occurrence of B given that A has already happened.
Sometimes, P (A/B) is also used to denote the probability of occurrence of A when B occurs.
Similarly, P (B/A) is used to denote the probability of occurrence of B when A occurs.

Note: If A and B are two independent events associated with a random experiment, then
P (A/B) = P (A) and P (B/A) = P (B) and vice – versa.

MULTIPLICATION THEOREMS ON PROBABILITY

Theorem 1: If A and B are two events associated with a random experiment, then

$\mathbf{P(A \cap B) = P(A)\,P(B/A)}$ if $P(A) \neq 0$

$\mathbf{P(A \cap B) = P(B)\,P(A/B)}$ if $P(B) \neq 0$

OR.

P (B/A) = $\frac{P(A \cap B)}{P(A)}$ and **P (A/B) =** $\frac{P(A \cap B)}{P(B)}$

Theorem 2: (Extension of Multiplication Theorem) If $A_1, A_2, \ldots A_n$ are n events related to a random experiment, then

$\mathbf{P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\,P(A_2/A_1)\,P(A_3 / A_1 \cap A_2) \ldots P(A_N / A_1 \cap A_2 \cap \ldots \cap A_{n-1})}$

where $P(A_N / A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ represents the conditional probability of the event A_1, given that the events $A_1 A_2 \ldots A_{n-1}$ has already happened.

EXAMPLES

1. A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and second is black?

Sol. Consider the following events:
A = getting a white ball in first draw, B = getting a black ball in second draw.
Required probability = Probability of getting a white ball in first draw and a black ball in second draw = $P(A \text{ and } B) = P(A \cap B) = P(A)\,P(B/A)$ ….(I) [by Multiplication Theorem]
Now, $P(A) = {}^{10}C_1 / {}^{25}C_1 = 10/25 = 2/5$ and P(B/A) = Probability of getting a black ball in second draw when a white ball has already been in first draw = ${}^{15}C_1 / {}^{24}C_1 = 15/24 = 5/8$ [as 24 balls are left after drawing a white ball in first-draw out of which 15 are black]
From (i), Required probability = $P(A \cap B) = P(A)\,P(B/A) = (2/5) \times (5/8) = 1/4$.

2. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one without replacement, find the probability of getting of white balls.

Sol. Let A, B, C, D denote events of getting a white ball in first, second, third and fourth draw respectively.

The required probability $P(A \cap B \cap C \cap D) = P(A)\ P(B/A)\ P(C/A \cap B)\ P(D/A \cap B \cap C)$ ….. (i)

Now, P(A) = Probability of drawing a white ball in first draw = 5/20 = 1/4.

$\therefore$ P(B/A) = 4/19.

Since the ball drawn is not replaced, therefore after drawing a white ball in second draw there are 18 balls left in the bag, out of which 3 are white.

$\therefore P(C/A \cap B) = 3/18 = 1/6$.

After drawing a white ball in third draw there are 17 balls left in the bag, out of which 2 are white.

$\therefore P(D/A \cap B \cap C) = 2/17$.

$\therefore$required probability = $P(A \cap B \cap C \cap D) = P(A)\ P(B/A)\ P(C/A \cap B)\ P(D/A \cap B \cap C)$

$= (1/4) \times (4/19) \times (1/6) \times (2/17) = 1/989$.

3. If A and B are two events such that P(A) = 0.5, P(B) = 0.6 and P(A ∪ B) = 0.8, find P(A/B) and P(B/A).

Sol. We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.5 + 0.6 - 0.8 = 0.3$

Now, $P(A/B) = [P(A \cap B)/P(B)] \Rightarrow P(A/B) = 0.3/0.6 = ½$

and $P(B/A) = [P(A \cap B)/P(A)] \Rightarrow P(B/A) = 0.3/0.5 = 3/5$.

4. A couple has 2 children. Find the probability that both are boys, if it is known that (i) one of the children is a boy; (ii) the older child is a boy.

Sol. Let B_i and G_i stand for i^{th} child be a boy and girl respectively, Then the sample space can be expressed as $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$.

Consider the following events:

A = both the children are boys; B = one of the children is a boy; C = the older children is a boy.

Then $A = \{B_1B_2\}$, $B = \{B_1G_2, B_1B_2, G_1B_2\}$ and $C = \{B_1B_2, G_1B_2\}$

So, $A \cap B = \{B_1B_2\}$ and $A \cap C = \{B_1B_2\}$

(i) Required probability = $P(A/B) = [P(A \cap B)/P(B)] = [(1/4)/(3/4)] = 1/3$.

(ii) Required probability = $P(A/C) = [P(A \cap C)/P(C)] = \{(1/4)/(2/4)] = ½$.

5. A coin is tossed twice and the four possible outcomes are assumed to be equally likely. If A is the event, 'both head and tail have appeared', and B the event, 'at most one tail is observed', find P(A) , P(B), P(A/B) and P(B/A).

Sol. Here S = {HH, HT, TH, TT}, A = {HT, TH} and B = {HH, HT, TH}. $\therefore A \cap B$ = {HT, TH}

Now, P(A) = n(A)/n(S) = 2/4 = ½, P(B) = n(B)/n(S) = ¾ and $P(A \cap B) = [n(A \cap B)/n(S)] = 2/4 = ½$.

$\therefore P(A/B) = [P(A \cap B)/P(B)] = [(1/2/(3/4)] = 2/3$ and $P(B/A) = [P(A \cap B)/P(A)] = [(1/2/(1/2)] = 1$

MULTIPLICATION THEOREMS FOR INDEPENDENT EVENTS

Theorem 1: If $A_1\ A_{2,\ldots}\ldots A_n$, are independent events associated with a random experiment, then

$$\mathbf{P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n) = P(A_1)\,P(A_2)\ldots\ldots P(A_n)}$$

For instance, If A and B are independent events associated with a random experiment, then

$$\mathbf{P(A \cap B) = P(A)\,P(B)}$$

i.e., the probability of simultaneous occurrence of two independent events is equal to the product of their probability.

Theorem 2: If A and B are independent events associated with a random experiment, then

(i) $\overline{A}$ **, B** (ii) **A,** $\overline{B}$ (iii) $\overline{A}$ **,** $\overline{B}$ **are also independent events.**

ADDITION THEOREM FOR INDEPENDENT EVENTS

Theorem: If $A_1\ A_{2,\ldots}\ldots A_n$, are n independent events associated with a random experiment. Then

$$\mathbf{P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n) = 1 - P(\overline{A}_1)\,P(\overline{A}_2)\ldots.\,P(\overline{A}_n)}$$

Baye's Theorem or Baye's Rule

Let S be the sample space and let E_1, E_2 …., E_n be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or …. Or E_n then

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}, \; i = 1, 2, \ldots, n$$

EXAMPLES

1. **Events E and F are independent. Find P(F) if P(E) = 0.35 and P(E $\cup$ F) = 0.6.**

Sol. We have $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\Rightarrow P(E \cup F) = P(E) + P(F) - P(E)\,P(F)$

$\Rightarrow P(E \cup F) = P(E) + P(F)\,(1 - P(E))$

$\Rightarrow 0.6 = 0.35 + P(F)\,(1 - 0.35)$

$\Rightarrow 0.25 = (0.65)P\,(F)$

$\Rightarrow P(F) = 5/13.$

2. **If P(A) = 0.4, P(B) = p, P(A $\cup$ B) = 0.6 and A and B are given to be independent events, find the value of p.**

Sol. Since A and B are independent events, therefore $P(A \cap B) = P(A)\,P(B)$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$\Rightarrow P(A \cup B) = P(A) + P(B)\,(1 - P(A))$

$\Rightarrow 0.6 = 0.4 + p(1 - 0.4)$

$\Rightarrow 0.2 = 0.6p \Rightarrow p = 1/3.$

3. A policeman fires four bullets on a dacoit. The probability that the dacoit will be killed by one bullet is 0.6. What is the probability that the dacoit is still alive?

Sol. Let A_i be the event that the dacoit is not killed by the 1st bullet. Then P(A) = 1 – 0.6 = 0.4.
Now, probability that the dacoit is still alive = $P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) P(A_2) P(A_3) P(A_4)$ [since all 4 shots are independent] = $(0.4)^4$ = 0.004096.

4. The probability that a teacher will give an unannounced test during any class meeting is 1/5. If a student is absent twice, what is the probability that he will miss at least one test?

Sol. Let E_i be the event that the student misses 1st test (i = 1, 2), then E_1 and E_2 are independent events such that $P(E_1) = 1/5 = P(E_2)$.
Required probability = $P(E_1 \cup E_2) = 1 – P(\bar{E}_1 \cup \bar{E}_2)$ [as E_1, E_2 are independent]
= 1 – [(1 – 1/5) (1 – 1/5)] = 9/25.

5. An article manufactured by a company consists of two parts X and Y. In the process of manufacturing of the part X, 9 out of 100 parts may be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part Y. Calculate the probability that the assembled product will not defective.

Sol. A = part X is not defective, B = Part Y is not defective.
Required probability = $P(A \cap B) = P(A) P(B) = (91/100) \times (95/100) = 0.8645$.

6. A bag contains 4 white and 2 black balls. Another contains 3 white and 5 black balls. If one ball is drawn from each bag, find the probability that
(i) Both are white (ii) one is white and one is black?

Sol. Consider the following events:
W_1 = Drawing a white ball from the first bag
W_2 = Drawing a white ball from the second bag
B_1 = Drawing a black ball from first bag
B_2 = Drawing a black ball from second bag
Clearly $(W_1) = 4/6$, $P(B_1) = 2/6$, $P(W_2) = 3/8$ and $P(B_2) = 5/8$

(i) P(both balls are white) = P[(white ball from 1st bag) and (white ball from 2nd bag)]
= $P(W_1 \cap W_2) = P(W_1) P(W_2)$ [as W_1 and W_2 are independent] = (4/6 x (3/8) = ¼.

(ii) P(one ball white and one ball black) = P[(black ball from 1st and white ball from 2nd bag) or white ball from 1st and black from 2nd bag)]
= $P[(B_1 \cap W_2) \cup (W_1 \cap B_2)]$
= $P(B_1 \cap W_2) + P(W_1 \cap B_2)$ [by ad(d) theorem for mutually exclusive events]
= $P(B_1) P(W_2) + P(W_1) P(B_2)$ [as B_1 & W_2; B_2 and W_1 are independent events]
= $(2/6) \times (3/8) + (4/6) \times (5/8) = 13/24$

7. A bag contains 5 white and 3 black balls. Four balls are successively drawn out without replacement. What is the probability that they are alternately of different colours?

Sol. Let W_i denote the event of drawing a white ball in i^{th} draw and B_i denote the event of drawing a black ball in i^{th} draw, where i = 1, 2, 3, 4.
Then, Required probability = $P[(W_1 \cap B_2 \cap W_3 \cap B_4) \cup (B_1 \cap W_2 \cap B_3 \cap W_4)]$
= $P(W_1 \cap B_2 \cap W_3 \cap B_4) + P(B_1 \cap W_2 \cap B_3 \cap W_4)$ [by ad(d) theorem]
= $P(W_1) P(B_2/W_1) P(W_3/W_1 \cap B_2) P(B_4 / W_1 \cap B_2 \cap W_3)$
+ $P(B_1) P(W_2 / B_1) P(B_3/B_1 \cap W_2 \cap B_3) P(W_4 / B_1 \cap W_2 \cap B_3)$ [by multiplication theorem]
= (5/8) x (3/7) x (4/6) x (2/5) + (3/8) x (5/7) x (2/6) x (4/5) = 1/7.

8. The odds against a husband who is 45 years old living till he is 70 are 7 : 5 and the odds against his wife who is now 36 living till she is 61 are 5: 3. Find the probability that
(i) the couple will be alive 25 years hence,
(ii) exactly one of them will be alive 25 years hence,
(iii) none of them will be alive 25 years hence,
(iv) at least one of them will be alive 25 years hence,

Sol. Let A be the event that the husband will be alive for 25 years hence and
B be the event that the wife will be alive 25 years hence.
Then A and B are independent events such that
P(A) = 5/(7 + 5) = 5/12 and P(B) = 3/(5 + 3) = 3/8.

(i) P (couple will be alive 25 years hence) = P(A and B) = $P(A \cap B)$ = P(A) P(B)
= (3/12) x (3/8) = 5/32

(ii) Exactly one of them will be alive 25 years hence in two mutually exclusive ways:
(i) Husband will be alive 25 years hence and wife will not, I.e., $A \cap \overline{B}$
(ii) Wife will be alive 25 years hence and husband will not, i.e., $\overline{A} \cap B$
$\therefore$ P(Exactly one will be alive 25 years hence) = P(I or II) = $P(A \cap \overline{B}) \cup (\overline{A} \cap B)$
= $P(A \cap \overline{B}) + P(\overline{A} \cap B)$ [as $A \cap \overline{B}$) $\overline{A} \cap B$ are mutually exclusive]
= $P(A)\,P(\overline{B}) + P(\overline{A})\,P(B)$ [as A, B are independent]
= (5/12) × (1 – 3/8) + (1 – 5/12) × (3/8) = (5 / 12) × (5/8) + (7/12) × (3/8)
= 46/96 = 23/48.

(ii) P(none of them will be alive 25 years hence)
= $P(\overline{A} \cap \overline{B}) = P(\overline{A})\,P(\overline{B})$ [as A, B are independent]
= (1 – 5/12) x (1 – 3/8)
= (7/12) x (5/8) = 35/96.

(iii) P(at least one of them will be alive 25years hence)
= $1 - P(\overline{A})\,P(\overline{B})$ [as A, B are independent]
= 1 – (1 – 5/12) (1 – 3/8) = 1 – 35/96 = 61/96.

9. In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?

Sol. Let E_1, E_2 and E_3 and A be the events defined as follows:
E_1 = the bolt is manufactured by machine A,
E_2 = the bolt is manufactured by machine B,
E_3 = the bolt is manufactured by machine C.
A = the bolt is defective.

Then $P(E_1)$ = Probability that the bolt drawn is manufactured by machine A = $\frac{25}{100}$,

$P(E_2)$ = Probability that the bolt drawn is manufactured by machine B = $\frac{35}{100}$

$P(E_3)$ = Probability that the bolt drawn is manufactured by machine C = $\frac{40}{100}$.

$P(A/E_1)$ = Probability that the bolt drawn is defective given that is manufactured by machine A = $\frac{5}{100}$.

Similarly, $P(A/E_2) = \frac{4}{100}$ and $P(A/E_3) = \frac{2}{100}$.

Now, required probability = Probability that the bolt is manufactured by machine B given that the bolt drawn is defective = $P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{35}{100} \times \frac{4}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}}$$

$$= \frac{140}{125 + 140 + 80} = \frac{140}{345} = \frac{28}{69}.$$