MATRICES

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A system of mn numbers arranged in a rectangular array of m rows and n columns is called an m by n matrix, which is written as m × n matrix.

Thus A = $\begin{bmatrix} a_{11} & a_{12} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2j} & ...a_{2n} \\ .... & .... & .... & . ... \\ a_{i1} & a_{i2} & ...a_{ij} & ...a_{in} \\ .... & .... & .... & . ... \\ a_{m1} & a_{m2} & a_{mj} & ...a_{mn} \end{bmatrix}$ is a matrix of order m x n. It has m rows and n columns.

Note: The element a_{ij} is the element in the ith row and jth column of A

TYPES OF MATRICES

Row Matrix: A matrix having a single row is called a row matrix e.g., [1 3 5 7].

Column Matrix: A matrix having a singe column is called a column matrix, e.g. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Square matrix: A matrix having n rows and n columns is called a square matrix of order n.

Diagonal matrix: A square matrix all of whose elements except those in the leading diagonal are zero is called a diagonal matrix.

Ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Scalar matrix: A diagonal matrix whose all the leading diagonal elements are equal is called a scalar matrix.

Ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Unit matrix: A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix or an identity matrix of order n and is denoted by I_n.

For example, unit matrix of order 3 is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Triangular matrix: A square matrix all of whose elements below the leading diagonal are zero is called an upper triangular matrix.

A square matrix all of whose elements above the leading diagonal are zero is called *a* ***lower triangular matrix***.

Thus $\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$ are upper and lower triangular matrices respectively

Transpose of a matrix: The transpose of a given matrix A is obtained by interchanging the rows and columns of the matrix A. Transpose of A is denoted by A' or A'.
Thus, if A = $[a_{ij}]_{mxn}$ Then A' = $[b_{ij}]_{nxm}$ Where $b_{ij} = a_{ji}$

Ex. If A = $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 7 & 0 \end{pmatrix}$ Then A' = $\begin{pmatrix} 1 & 2 \\ 3 & 7 \\ 5 & 0 \end{pmatrix}$

Properties of Transpose: Let A and B be two matrices. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$, A and B being of the same order.

(iii) $(kA)^T = kA^T$, k be any scalar (real or complex).

(iv) $(AB)^T = B^TA^T$, A and B being conformable for the product AB.

Conjugate of a Matrix: The conjugate of a given matrix A is obtained by replacing the elements of A by their corresponding complex conjugates. The conjugate of A is denoted by $\overline{A}$.

Thus, if A = $[a_{ij}]_{mxn}$ then $\overline{A} = [\overline{a}_{ij}]_{mxn}$

Ex: If A = $\begin{pmatrix} 1 & 1+i \\ 3 & 1-1 \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} 1 & 1 & -i \\ 3 & 1 & +1 \end{pmatrix}$

Tranjugate or Transposed Conjugate of a Matrix: The transpose conjugate of a given matrix is obtained by interchanging the rows and columns of the matrix obtained by replacing the element of A by their corresponding complex conjugate. The transpose conjugate of A is denoted by A* or by A^θ. Thus,

$A^* = (\overline{A'}) = (\overline{A})'$

Ex. If A = $\begin{pmatrix} 2 & 1+i & 0 \\ 3 & 2 & i \end{pmatrix}$ then A* = $\begin{pmatrix} 2 & 3 \\ 1-i & 2 \\ 0 & -i \end{pmatrix}$

Symmetric and skew-symmetric matrices: A square matrix A = $[a_{ij}]$ is said to be symmetric when $a_{ij} = a_{ji}$ for all i and j.
If $a_{ij} = -a_{ji}$ for all i and j so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix.

Examples of symmetric and skew-symmetric matrices are respectively.

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$$

Note: *Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix as:*

A = ½ (A + A') + ½ (A – A').

Hermitian Matrix: A square matrix A is said to be a Hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself i.e., $A^* = A \Rightarrow \overline{a_{ij}} = a_{ji}$, where $A = [a_{ij}]_{n \times n}$; $a_{ij} \in C$.

Ex : $\begin{pmatrix} 2 & 3+2i \\ 3-2i & 7 \end{pmatrix}$ $\begin{pmatrix} 3 & 2+i & 5i \\ 2-i & 0 & 2 \\ -5i & 2 & 7 \end{pmatrix}$

Skew Hermitian Matrix: A square matrix A is said to be skew-Hermitian, if

$$A^* = -A \Rightarrow \overline{a_{ij}} = -a_{ji}$$

Ex. $\begin{pmatrix} 2i & 5+4i \\ -5-4i & 0 \end{pmatrix}$ $\begin{pmatrix} 4i & 2-i & 3 \\ -2+i & 0 & 4 \\ -3 & -4 & -3i \end{pmatrix}$ are skew –Hermitian matrices.

Idempotent Matrix: A square matrix A, such that $A^2 = A$ is called idempotent matrix

Ex. $\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ is idempotent because

$$A^2 = A.\,A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 4+2-4 & -4-6+4 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{pmatrix}$$

$$A^2 = A.\,A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 4+2-4 & -4-6+4 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = A$$

Nilpotent Matrix: If A is a square matrix such that $A^m = 0$, where m is a positive integer, than A is called a ***nilpotent matrix***. If m is the least positive integer for which $A^m = 0$, then A is called to be nilpotent matrix of index m.

Ex. The matrix A = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$ is a nilpotent matrix of index 2, because $A^2 = A\,.\,A$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 1+2-3 & 2+4-6 & 3+6-9 \\ 1+2-3 & 2+4-6 & 3+6-9 \\ -1-2+3 & -2-2+6 & -3-6+9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Involutory Matrix: A square matrix A such that $A^2 = I$ is called involutory matrix.

Ex. The matrix A = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is involutory because $A^2 = A\,.\,A$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1.1+0.0 & 1.0+0(-1) \\ 0.1+(-1).0 & 0.0+(-1).(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Orthogonal Matrix: A square matrix A is said to be orthogonal matrix, if AA' = I = A' A.

Ex. The matrix $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ is orthogonal matrix, because

$$A\,A' = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \times \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Similarly A'A = 1

Unitary Matrix: A square matrix A is said to be unitary matrix, if AA* = I = A*A.

Ex. $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is an unitary matrix, because $AA^* = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} x = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$= \frac{1}{3}\begin{pmatrix} 1+1-i^2 & 0 \\ 0 & 1-i^2+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$ Similarly, A*A = I.

Singular Matrix: A square matrix A is called singular if $|A| = 0$.

Non singular Matrix: A square matrix A is called non-singular if $|A| \neq 0$.

Properties of Matrices:

Addition and subtraction of matrices: If A, B be two matrices of the same order, then their sum A + B is defined as the matrix each element of which is the sum of the corresponding elements of A and B

Thus $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

Similarly A – B is defined as a matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.

Thus $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \end{bmatrix}$$

Equal matrices: Two matrices A and B are said to be equal, written as A = B, if

- They are both of the same order i.e. have the same number of rows and columns, and
- The elements in the corresponding places of the two matrices are the same

1. Only matrices of the same order can be added or subtracted.
2. Addition of matrices is commutative,
   i.e. A + B = B + A.
3. Addition and subtraction of matrices is associative.
   i.e. (A + B) – C = A + (B – C) = B + (A – C).
4. A – B = A + (– B)
5. $A + B = A + C \Rightarrow B = C$
6. A + 0 = A = 0 + A

Multiplication of matrix by a scalar: The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.

$$\text{Thus } k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

The distributive law holds for such products, i.e. k (A + B) = kA + kB.
All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars. Therefore

- $(k + \ell)\,A = kA + \ell A$
- $(k\ell)\,A = k\,(\ell A) = \ell\,(kA)$
- $\ell A = A\ell,\ \ell \in R$ (or C)
- k (A + B) = kA + kB

Ex: Evaluate 3A – 4B, where

$$A = \begin{bmatrix} 3 & -4 & 6 \\ 5 & 1 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\text{We have} \quad 3A = \begin{bmatrix} 9 & -12 & 18 \\ 15 & 3 & 21 \end{bmatrix} \text{ and } 4B = \begin{bmatrix} 4 & 0 & 4 \\ 8 & 0 & 12 \end{bmatrix}$$

$$\therefore\ 3A - 4B = \begin{bmatrix} 9-4 & -12-0 & 18-4 \\ 15-8 & 3-0 & 21-12 \end{bmatrix} = \begin{bmatrix} 5 & -12 & 14 \\ 7 & 3 & 9 \end{bmatrix}$$

Multiplication of matrices: Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. Such matrices are said to be ***conformable*.**

For example, If $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{pmatrix}$, then A and B are conformable for the product AB such that $(AB)_{11}$ = (First row of A) (First column of B) = [2 1 3] $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ = 2*1 + 1*2 + 3*4 = 16 $(AB)_{12}$ = (First row of A) (Second column of B) = [2 1 3] $\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ = 2* – 2 + 1*1 + 3 * – 3 = – 12 etc. Thus AB = $\begin{pmatrix} 16 & -12 \\ 3 & -11 \\ 3 & -1 \end{pmatrix}$.

Properties of Matrix Multiplication:

- Matrix multiplication is associative i.e. if A, B, C are m x n, n x p and q x q matrices respectively, then (AB) C = A (BC)
- Matrix multiplication is not always commutative → AB $\neq$ BA
- Matrix multiplication is distributive over matrix addition i.e. A (B + C) = AB + AC where A, B, C are m x n, n x p, n x p matrices respectively
- The product of two matrices can be the null matrix while neither of them is the null matrix.
  For example, if $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ while neither A nor B is a null matrix.
- Two matrix are said to be commute if AB = BA.
  If AB = – BA, they are called Anti-commute.

Power of a matrix: If *A* be a square matrix, then the product *AA* is defined as A^2. Similarly, we define higher powers of *A*. i.e. $A.A^2 = A^3$, $A^2.A^2 = A^4$ etc.

Exam. Prove that $A^3 - 4A^2 - 3A + 11\,I = 0$, *where* $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

$$A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

$A^3 - 4A^2 - 3A + 11\,I$

$$\begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11, & 37-28-9+0, & 26-20-6+0 \\ 10-4-6-0, & 5-16+0+11, & 1-4+3+0 \\ 35-32-3+0, & 42-36-6+0, & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Adjoint of A Matrix: The determinant of the square matrix

$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\Delta \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The matrix formed by the cofactors of the elements in Δ is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$.

Then, the transpose of this matrix, i.e. $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

is called the *adjoint of the matrix A* and is written as *Adj. A.*
Thus the Adjoint *of A is the transposed matrix of cofactors of A.*

Inverse of a matrix:
Let A be a square matrix of order n. If there exists a matrix B, such that
A. B = B. A = I, then B is called the inverse of the matrix and denoted by A^{-1}.

- The Inverse of a Square Matrix A exists if and only if A is non-singular (i.e., det A ≠ 0).
- The Inverse a Square Matrix is unique.
- If A is a non-singular matrix then $A^{-1} = \dfrac{\text{adj } A}{\det A}$

Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Example: If $A = \begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$, then A + B = _______ and A – B = _____

Solution: $A + B = \begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2+1 & 0+(-2) \\ -3+0 & 4+4 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 8 \end{bmatrix}$

$A - B = \begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2-1 & 0-(-2) \\ -3+0 & 4-4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -3 & 0 \end{bmatrix}$

Example: Evaluate $\Delta = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$

Operating $r_1 \to r_1 + r_2 + r_3$ we get,

$\Delta = \begin{bmatrix} 1+\omega+\omega^2 & \omega+\omega^2+1 & \omega^2+1+\omega \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} = 0 \quad [\because (1+\omega+\omega^2) = 0]$

Example: Find the inverse of the matrix

Solution: $\det A = 1\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} - 2\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} + 5\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

= – 2 – 4 + 10 = 14

$\Rightarrow \det A \neq 0$

$\therefore$ A is non – singular

Let A_{ij} denote the co-factor of a_{ij} of the matrix A.

$\therefore A_{11} = (-1)^{1+1}\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} = -2,$

$A_{12} = (-1)^{1+2}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = -2,$

$A_{13} = (-1)^{1+3}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = 2,$

$A_{21} = (-1)^{2+1}\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} = 1,$

$A_{22} = (-1)^{2+2}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 1,$

$A_{31} = (-1)^{3+1}\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} = 3,$

$A_{32} = (-1)^{3+2}\begin{bmatrix} 1 & 5 \\ 3 & 4 \end{bmatrix} = 11,$

$$A_{33} = (-1)^{3+3}\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix} = -5,$$

$$\therefore \text{Matrix of co-factors} = \begin{bmatrix}-2 & -2 & 2\\1 & -3 & 1\\3 & 11 & -5\end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix}-2 & 1 & 3\\-2 & -3 & 11\\2 & 1 & -5\end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{1}{4}\begin{bmatrix}-2 & 1 & 3\\-2 & -3 & 11\\2 & 1 & -5\end{bmatrix}$$

Example: If $A = \begin{Vmatrix}\text{Cosh}\alpha & \text{Sinh}\alpha\\\text{Sinh}\alpha & \text{Cosh}\alpha\end{Vmatrix}$, find A^{-1}

Solution: $\det A = \begin{Vmatrix}\text{Cosh}\alpha & \text{Sinh}\alpha\\\text{Sinh}\alpha & \text{Cosh}\alpha\end{Vmatrix}$

$= \cos h^2\alpha - \sinh^2\alpha = 1 \neq 0.$ $\quad\therefore A^{-1}$ can be computed

RANK OF A MATRIX:

If we select any *r* rows and *r* columns from any matrix *A*, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the *minor of A of order r.* Clearly there will be a number of different minors of the same order, got by deleting different rows and columns form the same matrix.

Def. *A matrix is said to be of rank r when*

(i) it has at least one non-zero minor of order r,

and (ii) every minor of order higher than r vanishes.

Briefly, *the rank of a matrix is the largest order of any non-vanishing minor of the matrix.*

If a matrix has a non-zero minor of order *r*, its rank is $\geq r$.

If all minors of a matrix of order *r* + 1 are zero, its rank is $\leq r$.

Note:

1. Since the rank of every non-zero matrix is $\geq$ 1, we agree to assign the rank, zero to every null matrix.
2. Every (r + 1) rowed minor of a matrix can be expressed as the sum of its r-rowed minors Therefore if all the r-rowed minors of a matrix are equal to zero, then obviously all its (r + 1) rowed minors will also be equal to zero.

Important: **The following two simple results will help us very much in finding the rank of a matrix.**

(i) The rank of a matrix is $\leq r$, if all (r + 1)-rowed minors of the matrix vanish

(ii) The rank of a matrix is $\geq$ r, if there is at least one r-rowed minor of the matrix which is not equal to zero.

Examples:

1. Let A = I_3 = $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ be a unit matrix of order 3.

   We have $|A|$ = 1 Therefore A is a non-singular matrix Hence rank A = 3.In particular, the rank of a unit matrix of order n is n

2. Let A = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since A is a null matrix, therefore rank A = 0

3. Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}$ We have $|A|$ = 1 (6 – 8) – 2(4 – 4) + 3(4 – 3) = 2 + 1 = 3. Thus A is a non-singular

Therefore rank A = 3.

4. Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$. We have $|A|$ = 1 (24 – 25) – 2 (18 – 20) + 3 (15 – 16) = 0.

Therefore the rank of A is less than a 3. Now there is at least one minor of A or order 2, namely $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ which is not equal to zero Hence rank A = 2.

5. Let A = $\begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$ we have $|A|$ = 0, since the first two columns are identical Also each 2 – rowed minor of A is equal to zero. But A is not a null matrix. Hence rank A = 1

6. Let A = $\begin{pmatrix} 2 & 4 & 3 & 2 \\ 3 & 5 & 1 & 4 \end{pmatrix}$ Here we see that there is at least one minor of A of order 2 i.e., $\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ which is not equal to zero Also there is no minor of A of order greater that 2 Hence rank of A = 2

Echelon form of matrix: A matrix A is said to be Echelon from if

(i) Every row of A which has all its entries 0 occurs below every row, which has a non-zero entry.

(ii) The first non-zero entry in each non-zero row is equal to 1.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Important result:

The rank of a matrix in Echelon from is equal to the number of non-zero rows of the matrix.

Ex Find the rank of the matrix $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Sol. The matrix A has zero row. We see that it occurs below every non-zero row. Further the number of zero before the first non-zero element in the first row is one. The number of zeros before the first non-zero element in the second row it two. Thus the number of zeros before the first non-zero element in any row is less than the number of such zeros in the next row. Thus the matrix A is in Echelon from → rank A = the number of non-zero rows of A = 2.

Examples

1. *Find the rank of each of the following matrices:*

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}$

Sol. **(i)** Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ We have $|A| = 1(2 – 0) + 3(2 – 0)$, expanding along the first row

= 2 – 8 + 6 = 0.

But there is at least one minor of order 2 of the matrix A, namely $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ which is not equal to zero.

Hence the rank A = 2.

(ii) Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}$ Here there is at least one minor of order 2 of the matrix A, namely $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ which is not equal to 0. Also there is not minor of the matrix A of order greater than 2. Hence rank A = 2.

2. Show that the rank of a matrix every element of which is unity is 1.

Sol. Let A denote a matrix every element of which is unity. All the 2-rewed minors of A obviously vanish but A is a non-zero matrix. Hence rank A = 1.

3. A is a non-zero column and B a non-zero row matrix, show that rank (AB) = 1.

Sol. A = $\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \ldots\ldots \\ a_{ml} \end{pmatrix}$ and B = $(b_{11}\ b_{12}\ b_{13}\ \ldots\ldots\ b_{in})$

be two non-zero column and row matrices respectively.

We have AB = $\begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & \ldots\ldots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & \ldots\ldots & a_{21}b_{1n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{ml}b_{11} & a_{ml}b_{12} & a_{ml}b_{13} & \ldots\ldots & a_{ml}b_{1n} \end{pmatrix}$

Since A and B are non-zero matrices therefore the matrix AB will also be non-zero The matrix AB will have at least one non zero element obtained by multiplying corresponding non-zero elements of A and B All the two-rowed minors of A obviously vanish. But A is a non-zero matrix. Hence rank AB = 1.

4. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ find the ranks of U and U^2

Sol. The matrix U is the Echelon form. → rank U = the number of non-zero rows of U = 3.

Now $U^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Elementary Operations: An elementary operation is said to be row (or column) operation if it is applied to rows (or columns). There are three types of elementary operations:

Type 1: The interchanging of i^{th} row (or columns) of the matrix, this operation is denoted by R_i <--> R_j or (C_i <--> C_j) or by R_{ij} (C_{ij})

Type 2: If ith row (or column) of the matrix is multiplied by scalar $K(K \neq 0)$, then this row (or column) operation is denoted by R_i <--> kR_j (C_i <--> C_j)

Type 3: If k times of the elements of the i^{th} row (or column) are added to the corresponding elements of j^{th} row (or column), then this row (column) operation is denoted by $R_J \rightarrow R_J$ or kR_i (or $C_j \rightarrow C_j + kC_j$

Examples:

1. If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $E_{12} - E_{21}$ is equal to:

(a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 01 & 0 \\ 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ (d) None of these

Sol. Given that = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Therefore $E_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_1 < \Rightarrow R_{12}$ $E_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $R_1 <\Rightarrow R_2$

$E_{12} - E_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Therefore the correct answer is (b).

2. If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $2E_{12}$ (2) + 3 E_{21} is equal to:

(a) $\begin{pmatrix} 2 & 7 \\ 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 2 \\ 2 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 7 \\ 3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 4 \\ 3 & 1 \end{pmatrix}$

Sol. $E_{12} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $R_1 \rightarrow R_1 + 2R_2$ $E_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $R_1 \Leftrightarrow R_2$

Therefore $2E_{12}$ (3) + $3E_{21}$ = $2 \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 3 & 2 \end{pmatrix}$.

Therefore the correct answer is (c).

Elementary matrices: An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.

Examples of elementary matrices obtained from

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are } R_{23} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23};\ kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix};\ R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equivalent matrix: Two matrices A and b are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol ~ is used for equivalence.

IMPORTANT FACTS

1. The matrix obtained by applying a row operation on a given matrix is same as the matrix obtained by pre-multiplication of the given matrix by the corresponding elementary matrix.

   For Example: Let A = $\begin{pmatrix} 3 & 4 \\ 8 & 9 \end{pmatrix}$ Let us operate R_{12}, then A ~ $\begin{pmatrix} 8 & 9 \\ 3 & 4 \end{pmatrix}$ ….. (1)

   Now let 1 = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then by R_{12} we get elementary matrix $E_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Now E_{12}. A = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 8 & 9 \\ 3 & 4 \end{pmatrix}$

   Which is same as seen in (1).

2. The matrix obtained by applying a column operation on a given matrix is same as the matrix obtained by post-multiplication of the given matrix by the corresponding elementary matrix.

3. The effect of a row operation on a product of two matrices is equivalent to the effect of the same row operation applied only to pre-factor.

4. The effect of a column operation on a product of two matrices is equivalent to the effect of the same column operation applied only to the post-factor.

SYSTEM OF LINEAR EQUATIONS

Consider the following system of m linear equations in n unknown $x_1, x_2, \ldots. x_n$.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots.. - a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots.. - a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots.. - a_{mn}x_n = b_m \end{array}\right\}$$

The above system, of equations can be written as AX = B.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots\ldots. & a_{1n} \\ a_{21} & a_{12} & \ldots\ldots. & a_{1n} \\ \ldots. & \ldots\ldots & \ldots\ldots. & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots. & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots.. \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots.. \\ b_n \end{pmatrix}$$

If B = 0 i.e., $b_1 = b_2 = b_3 \ldots. b_n = 0$, then the system of equations is called **homogeneous**

If B $\neq$ 0, then the system of equations is **non-homogeneous.**

SOLUTION OF A NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

There are three methods of solving a system of linear equations:

(i) **Matrix method**

(ii) **Rank method**

(iii) **Determinant method (Cramer's Rule)**

Matrix method: For the system of m-linear equations in n unknowns (variables) represented by AX = B.

(i) If $|A| \neq 0$, then the system is consistent and has a **unique solution** given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and (adj A) B = 0, then the system is consistent and has **infinitely many solutions**.

(iii) If $|A| = 0$ and (adj A) $B \neq 0$, then the system **is inconsistent (no solution)**.

Rank method: For the system of m-linear equations in n unknowns (variables) represented by AX = B.
If m > n, then

(i) If r (A) = r (A : B) = n, then system of linear equations has a unique solution.

(ii) If r (A) = r (A : B) = r < n, then system of linear equations is consistent and has infinite number of solutions. In fact, in this case (n – r) variables can be assigned arbitrary values.

(iii) If r (A) ≠ r (A : B), then the system of linear equations is inconsistent i.e. it has no solution.

If m < n and r (A) = r (A : B) = r, then there are infinite number of solutions.

Determinant method: For a system of 3 simultaneous linear equations in three unknowns.

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by
$x = \frac{D_1}{D'}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D = 0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then the given system of equations is inconsistent.

SOLUTION OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

A homogeneous system of linear equations, AX = O is never inconsistent, as it always have a trivial solution i.e. X = 0.

Matrix method:

Step I: If $|A| \neq 0$, then "the system is consistent with unique solution x = y = z = 0." (trivial solution)

Step II: If $|A| = 0$, then system of equations has infinitely many solutions.
To find these solutions proceed as follows:
Put z = k (any real number) and solve any two equations for x and y in terms of k. The values of x and y so obtained with z = k give a solution of the given system of equations.

Rank method:

If r (A) = n = number of variables, then AX = O has a unique solution X = 0 i.e. $x_1 = x_2 = \ldots = 0$. (trivial solution)
If r (A) = r < n (= number of variables), then the system of equations has an infinite number of solutions.

EXAMPLE

1. The rank of the matrix A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 10 & 18 \end{pmatrix}$ is :

(a) 0 (b) 1 (c) 2 (d) 3

Sol. Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 10 & 18 \end{pmatrix}$ Therefore

$|A| = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 10 & 18 \end{pmatrix} = 1\ (54 - 40) - 2(36 - 16) + 3\ (20 - 12) = -2 \neq 0.$

Therefore p (A) ≥ 3…..(i). The matrix A does not posses any minor of order 4.

Therefore p(A) ≥ 3 …..(ii) From (i) and (ii) p (A) = 3. Therefore the correct answer is (d)