DIFFERENTIABILITY

A function *f(x)* is said to be derivable at x = c $\in$ D(f) if $\underset{h\to 0}{\text{Lt}}\frac{f(c+h)-f(c)}{h}$ (called derivative $f'(c)$ at c) exist (finitely).

(i) **Left derivative at x = c**

$$Lf'(c) = \underset{h\to 0^-}{\text{Lt}}\frac{f(c-h)-f(c)}{-h}$$

(ii) **Right derivative at *x* = *c***

$$Rf'(c) = \underset{h\to 0^+}{\text{Lt}}\frac{f(c+h)-f(c)}{h}$$

Remark: From the above definitions, it follows that the derivative of a function *f* at *c* exists iff the left and right derivatives both exist separately at that point and are equal.

$$Lf'(c) = Rf'(c) = f'(c)$$

In other words, Derivative: Let *y* = *f* (*x*) be a function of *x* and δx be an increment in the value of *x* and δy be the corresponding increment in the value of the function y, then

$\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta y}$ (if it exists) is called the derivative of *y* with respect to *x* and is denoted by $\frac{dy}{dx}$ i.e $\frac{dy}{dx} = \underset{\delta\to 0}{\text{Lt}}\frac{\delta y}{\delta x}$.

THEOREM: If a function is finitely derivable at a point, then it is also continuous at that point; but the converse is not true.

Proof: - For let f (x) be derivable at x = a, then f ' (a) exists, hence

$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists and let it be equal to f ' (a).

Then $\lim_{h\to 0}$ [f (a + h) – f (a)] = $\lim_{h\to 0}$ f ' (a).

$\therefore$ $\lim_{h\to 0}$ f (a + h) = f (a) or equivalently $\lim_{z\to a}$ f (x) = f (a).

Hence the function is continuous at x = a.

Conversely, let f (x) = $|x|$, is continuous at x = 0 but it is not derivable at x = 0.

EXAMPLES

11. Let f(x) = $x^2 \sin\frac{1}{x}$ for x $\neq$ 0, f (0) = 0. then

(A) f (x) is continuous and derivable at x = 0

(B) f (x) is continuous and not differentiable at x = 0.

(C) f (x) is neither continuous nor differentiable at x = 0

(D) none of these

Sol. **Continuity at x = 0 :**

L $\underset{x\to 0}{\text{Lt}}$ f(x) = $\underset{h\to 0}{\text{Lt}}$ f(0 – h) = $\underset{h\to 0}{\text{Lt}}$ $(0-h)^2 \sin \frac{1}{0-h}$ = $\underset{h\to 0}{\text{Lt}}$ $= -h^2 \sin \frac{1}{h}$ = 0 x [finite quantity] = 0

R. $\underset{x\to 0}{\text{Lt}}$ f(x) = $\underset{h\to 0}{\text{Lt}}$ f(0 + h) = $\underset{h\to 0}{\text{Lt}}$ $(0+h)^2 \sin \frac{1}{0+h}$ = $\underset{h\to 0}{\text{Lt}}$ $h^2 \sin \frac{1}{h}$ = 0

Also f (0) = 0.

∴ L. $\underset{x\to 0}{\text{Lt}}$ f(x) = R. $\underset{x\to 0}{\text{Lt}}$ f(x) = f(0) .

∴ f(x) is continuous at x = 0.

Derivability at x = 0:

$$\text{L. } f'(0) = \underset{h\to 0}{\text{Lt}} \frac{f(0-h)-f(0)}{-h} = \underset{h\to 0}{\text{Lt}} \frac{(-h)^2 \sin\frac{1}{(-h)} - 0}{-h} = \underset{h\to 0}{\text{Lt}}\ h \sin \frac{1}{h} = 0 \text{ x finite quantity} = 0$$

$$\text{R. } f'(0) = \underset{h\to 0}{\text{Lt}} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{\text{Lt}} \frac{h^2 \sin\frac{1}{h} - 0}{h} = \underset{h\to 0}{\text{Lt}}\ h \sin \frac{1}{h} = 0$$

∴ L. f ' (0) = R. f ' (0) .

∴ f (x) is derivable at x = 0. **The correct answer is (A).**

12. The function f (x) = $|x-1| + |x-2|$ is not derivalble at

(A) x = 0 (B) x = 1 (C) x = 3 (D) x = 1, x = 2

Sol. The given function is f(x) = $|x-1| + |x-2|$ ∴ f(x) = $\begin{cases} (1-x)+(2-x) = 3-2x, & 0 \le x < 1 \\ (x-1)+(2-x) = 1, & 1 \le x < 2 \\ (x-1)+(x-2) = 2x-3, & 2 \le x \le 3 \end{cases}$

Clearly the function is derivable for $0 \le x < 1$, $1 < x < 2$, $2 < x \le 3$.

∴ Let us consider the points x = 1 and x = 2

At x = 1. L. f'(1) = $\underset{h\to 0}{\text{Lt}} \frac{f(1-h)-f(1)}{-h} = \underset{h\to 0}{\text{Lt}} \frac{3-2(1-h)-f(1)}{-h} = \underset{h\to 0}{\text{Lt}} (-2) = -2$

R. f ' (1) = $\underset{h\to 0}{\text{Lt}} \frac{f(1+h)-f(1)}{h} = \underset{h\to 0}{\text{Lt}} \frac{1-1}{h} = 0.$

∴ L f '(1) ≠ R. f ' (1) .

∴ f(x) is not derivable at x = 1.

At x = 2. L. f '(2) = $\underset{h\to 0}{\text{Lt}} \frac{f(2-h)-f(2)}{-h} = \underset{h\to 0}{\text{Lt}} \frac{1-1}{h} = 0$

R. f '(2) = $\frac{1-1}{h}$ $\frac{f(2+h)-f(2)}{h} = \underset{h\to 0}{\text{Lt}} \frac{2(2+h)-3-1}{h} = \underset{h\to 0}{\text{Lt}} (2) = 2$

∴ L. f '(2) ≠ R. f ' (2) .

∴ f (x) is not derivable at a x = 2. **The correct answer is (D).**

DIFFERENTIATION: - STANDARD RESULTS

Sr no	Function	Differentiation(dy/dx)	Remarks
1	$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$y = u^n, \frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$
2	$y = e^x$	$\frac{dy}{dx} = e^x$	$y = e^u, \frac{dy}{dx} = e^u \frac{du}{dx}$
3	$y = \log x$	$\frac{dy}{dx} = 1/x$	$y = \log u, \frac{dy}{dx} = 1/u \frac{du}{dx}$
4	$y = a^x$	$\frac{dy}{dx} = a^x \log a$	$y = a^u, \frac{dy}{dx} = a^u \log a. \frac{du}{dx}$
5	$y = \sin x$	$\frac{dy}{dx} = \cos x$	$y = \sin u, \frac{dy}{dx} = \cos u. \frac{du}{dx}$
6	$y = \cos x$	$\frac{dy}{dx} = -\sin x$	$y = \cos u, \frac{dy}{dx} = -\sin u. \frac{du}{dx}$
7	$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	$y = \tan u, \frac{dy}{dx} = \sec^2 u. \frac{du}{dx}$
8	$y = \cot x$	$\frac{dy}{dx} = -\text{cosec}^2 x$	$y = \cot u, \frac{dy}{dx} = -\text{cosec}^2 u. \frac{du}{dx}$
9	$y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$	$y = \sec u$ → Apply $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
10	$y = \text{cosec}\, x$	$\frac{dy}{dx} = -\text{cosec}\, x \cot x$	----- Do as above -----
11	$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	----- Do as above -----
12	$y = \cos^{-1} x$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$	----- Do as above -----
13	$y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$	----- Do as above -----
14	$y = \cot^{-1} x$	$\frac{dy}{dx} = -\frac{1}{1+x^2}$	----- Do as above -----
15	$y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert > 0$	----- Do as above -----
16	$y = \text{cosec}^{-1} x$	$\frac{dy}{dx} = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert > 0$	----- Do as above -----

SOME FUNDAMENTAL THEOREMS

Let u, v, w….. be functions of x whose derivatives exist.

1. Differential coefficient of constant is zero, i.e., $\frac{d}{dx}(k) = 0$

2. $\frac{d}{dx}(ku) = k\frac{du}{dx}$

3. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

4. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

5. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $\frac{DN' - ND'}{D^2}$

6. If y = f (t) and t = ϕ (x), then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

7. If u = f (y), then $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = f'(y)\frac{du}{dx}$

8. $\frac{dy}{dx}.\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{dx/dy}$

DIFFERENTIATION OF ONE FUNCTION WITH RESPECT TO ANOTHER FUNCTION

Let y = f (x), z = ϕ(x) and

Then derivative of f (x) w.r.t, ϕ(x), i.e., y with respect to z is:

$$\frac{dy}{dz} = \frac{dy}{dx} \div \frac{dx}{dz}$$

LOGARITHMIC DIFFERENTIATION

If $y = [f_1(x)]^{f_2(x)}$ or y = $f_1(x)\ f_2(x)\ f_3(x)$ …. or $y = \frac{f_1(x).f_2(x)..}{\phi_1(x).\phi_2(x)..}$

Then it will be convenient to take log of both sides before performing differentiation

IMPLICIT FUNCTION

f(x, y) = c. Differentiate each term w.r.t. x and note that $\frac{d}{dx}(\phi(y)) = \frac{d}{dy}(\phi(y))\frac{dy}{dx}$

For example, $x^3 + y^3 - 3axy = 0$. Differentiating w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left(1.y + x\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (x^2 - ay) + (y^2 - ax)\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ay}$$

BY THE HELP OF PARTIAL DIFFERENTIATION

If f(x, y) = c, then we can find $\frac{dy}{dx}$ by the help of partial differentiation as under

$\frac{dy}{dx} = -\frac{f_x}{f_y}$ where,

f_x is differential coefficient of f(x, y) w.r.t. x treating y as constant.

Similarly f_y is differentiation of f(x, y) w.r.t. y treating x as constant.

For example, if $f(x, y) = x^3 + y^3 - 3axy = 0$, then $f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$.

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x^2 - ay}{y^2 - ax}$$

TRIGONOMETRY FORMULAE

Sometimes by a trigonometric transformation the derivatives of various functions can be calculated in much simpler way than otherwise. The following trigonometric expansions and formulae are to be remembered.

1 i $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$

ii $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$

iii $2\tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2}$

iv $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1 - y^2} \pm y\sqrt{1 - x^2}\right]$

v $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{1 - x^2}\sqrt{1 - y^2}\right]$

2. $\sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$

3. $\sin^{-1} x = \text{cosec}^{-1} (1/x)$; $\tan^{-1} x = \cot^{-1} (1/x)$

4. $\sin^{-1}\cos x = \sin^{-1} \sin\left(\frac{1}{2}\pi - x\right) = \frac{1}{2}\pi - x$

5. $\tan^{-1} (\tan \theta) = \theta$, $\sin^{-1} (\sin \theta) = \theta$, $\cos^{-1} (\cos \theta) = \theta$

6 $\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2(x/2)}{2\cos^2(x/2)}=\tan^2\frac{x}{2}$; $\frac{1+\cos x}{1-\cos x}=\cot^2\frac{x}{2}$;

$\frac{1-\cos x}{\sin x}=\frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}=\tan\frac{x}{2}$; $\frac{1+\cos x}{\sin x}=\cot\frac{x}{2}$

7. $\sqrt{1\pm\sin x}$ = cos (x / 2) ± sin (x / 2),

8. tan A ± tan B = $\frac{\sin A\cos B\pm\cos A\sin B}{\cos A\cos B}=\frac{\sin(A\pm B)}{\cos A\cos B}$

9. $\tan\left(\frac{1}{4}\pi+\theta\right)=\frac{1+\tan\theta}{1-\tan\theta}$; $\tan\left(\frac{1}{4}\pi-\theta\right)=\frac{1-\tan\theta}{1+\tan\theta}$

10. $\cos x = 2\cos^2(x/2) - 1 = 1 - 2\sin^2(x/2) = \cos^2(x/2) - \sin^2(x/2)$

11 sin x = $\frac{2\tan(x/2)}{1+\tan^2(x/2)}$ 12. cos x = $\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$ 13 tan x = $\frac{2\tan(x/2)}{1-\tan^2(x/2)}$

14 $\sin 3x = 3\sin x - 4\sin^3 x$ 15 $\cos 3x = 4\cos^3 x - 3\cos x$

SUBSTITUTION FOR DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

on	on
$a^2 + x^2$	x = a tan θ or a cot θ
$a^2 - x^2$	x = a sin θ or a cos θ
$x^2 - a^2$	x = a sec θ or a cosec θ
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	x = a cos 2θ
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$\alpha\cos^2\theta + \beta\sin^2\theta$

SUCCESSIVE DIFFERENTIATION

Let y = f (x). Then if f (x) is differentiable then, $y_1 = \frac{dy}{dx} = f'(x)$ is also differentiable function of x.

The derivate of $\frac{dy}{dx}$ is denoted by $\frac{d^2y}{dx^2}$ or y_2 or f " (x) is also a differentiable function x.

Thus, we can differentiate f (x) any number (positive integer) of times.

If n is a positive integer then nth derivation of y or f is denoted by y_n or $\frac{d^ny}{dx^n}$ or $y^{(n)}$ or $D^n y$.

Thus $y_n = y^{(n)} = \frac{d^n y}{dx^n} = f^n(x) = D^n y$.

Leibnitz's Theorem: If u and v are any two functions of x such that their desired differential coefficients exist, then the nth differential coefficient of uv is given by

$D^n (uv) = (D^n u).v + {}^nC_1 (D^{n-1} u). (Dv) + {}^nC_2 (D^{n-2} n). (D^2 v) + \ldots\ldots + u (D^n v)$

Methods of Successive Differentiation

- By the use of **standard results** given above.
- By **decomposition into a sum:** Sometimes standard results given earlier can be used by decomposing the given functions as the sum or difference of suitable functions with the help of algebraic or trigonometric formulae.
- By the use of **partial fractions:** If numerator and denominator of a function are both rational, integral algebraic functions, then results given can be used by decomposing the given function into partial fractions.
- By **Leibnitz's Theorem:** To find the nth derivate of the product of two functions, Leibnitz's theorem is useful.

SOME IMPORTANT Nth DERIVATIVE RESULTS

FUNCTION y = f(x)	$\mathbf{y_n}$
$y = a^x$	$a^x (\log a)^n$
$y = (ax + b)^m$, $m \in I$,	Case 1 - **m** > n $\frac{m!}{(m-n)!}(ax + b)^{m-n} a^n$ Case 2 - **m = n** $n!\, a^n$ Case 3 - **m < n** 0
$y = \frac{1}{ax+b}$	$y_n = (-1)^n \underline{\lfloor n} (ax + b)^{-n-1} a^n$
$y = \log (ax + b)$	$y_n = (-1)^{n-1} \underline{\lfloor n-1}\, a^n (ax + b)^{-n}$
$y = \sin (ax + b)$	$y_n = a^n \sin\left(ax + b + n\frac{\pi}{2}\right)$
$y = \cos (ax + b)$	$y_n = a^n \cos\left(ax + b + n\frac{\pi}{2}\right)$

Proofs:

1. nth order derivative of a^x

Let $y = a^x$. Then $y_1 = a^x \log a$, $y_2 = a^x (\log a)^2$, $y_3 = a^x (\log a)^3$

So, in general $y_n = a^x (\log a)^n$. Thus $D^n (a^x) = a^x (\log a)^n$

2. nth order derivative of sin (ax + b)

Let y = sin (ax + b). Then $y_1 = a\cos(ax+b)$ or $y_1 = a\sin\left(\frac{\pi}{2}+ax+b\right)$

→ $y_2 = a^2\cos\left(\frac{\pi}{2}+ax+b\right) = a^2\sin\left(2\,\frac{\pi}{2}+ax+b\right)$.

Thus, $D^n\{\sin(ax+b))\} = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$.

In particular, when a = 1, b = 0, $D^n(\sin x) = \sin\left(\frac{n\pi}{2}+x\right)$

3. nth order derivative of $(ax + b)^m$

$y_2 = m(m-1)(ax+b)^{m-2}a^2$; $y_3 = m(m-1)(m-2)(ax+b)^{m-3}a^3$

Thus, $D^n\{ax+b)^m\} = m(m-1)(m-2)\ldots(m-(n-1))(ax+b)^{m-n}a^n$

This is the general formula for the nth order derivative of $(ax + b)^m$ for all values of m.

Now three cases arise -

Case I: When $m \in N$ and m > n. In this case

$D^n\{(ax+b)^m\} = m(m-1)(m-2)\ldots m-(n-1))(ax+b)^{m-n}a^n = \frac{m!}{(m-n)!}(ax+b)^{m-n}a^n$.

Case II: When $m \in N$ and m = n. In this case $D^n\{(ax+b)^n\} = n!\,a^n$.

Case III: When $m \in N$ and m < n. In this case $D^n\{(ax+b)^m\} = 0$.

Putting m = – 1, we get $D^n\{(ax+b)^{-1} = (-1)(-2)(-3)\ldots(-1-(n-1)(ax+b)^{-1-n}a^n$

$= \frac{(-1)^n\,1.2.3\ldots n}{(ax+b)^{n+1}}a^n = \frac{(-1)^n\,n!}{(ax+b)^{n+1}}a^n$.

Thus, $D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n\,n!}{(ax+b)^{n+1}}a^n$

In Particular, putting a = 1, b = 0, we get $D^n\left(\frac{1}{x}\right) = \frac{(-1)^n\,n!}{x^{n+1}}$

<u>EXAMPLES</u>

1. If y = tan x, prove that $y_2 = 2yy_1$

Sol. We have y = tan x

$\therefore \frac{dy}{dx} = \sec^2 x$ or $y_1 = \sec^2 x$

$\Rightarrow \frac{d}{dx}(y_1) = \frac{d}{dx}(\sec^2 x)$

$\Rightarrow y_2 = 2\sec x\,\frac{d}{dx}(\sec x) = 2\sec x.\sec x\tan x = 2\tan x.\sec^2 x$

$\Rightarrow y_2 = 2yy_1$ [$\because y = \tan x$ and $y_1 = \sec^2 x$]

2. If $y = x^3 \sin x$, then y_{20} = ?

(a) $x^3 \sin x - 60 x^2 \cos x - 1140 x \sin x + 6840 \cos x$

(b) $x^3 \sin x - 60 x^2 \cos x + 1140 x \sin x - 6840 \cos x$

(c) $x^3 \sin x + 60 x^2 \cos x + 1140 x \sin x + 6840 \cos x$

(d) none of these

Sol. Using Leibnitz Theorem, considering sin x as 1st function, we have

$y_{20} = \sin (x + 20\pi/2).x^3 + 20.\sin (x + 19\pi/2).\ 3x^2$

$+ \frac{1}{2}.\ 20.19.\ \sin (x + 18\pi/2).\ 6x + (1/6).\ 20.19.19.\sin (x + 17\pi/2).6$

$= x^3 \sin x - 60 x^2 \cos x - 1140 x \sin x + 6840 \cos x$. **Answer: (a)**

3. If y is a polynomial of degree n with its leading coefficient 2, then $D^{n-1}(y)$ =?

(a) 2 (n!) (b) 2 (n!) x (c) 2 (n – 1) ! x (d) none of these

Sol. Let $y = 2x^n + a_1 x^{n-1} + \ldots\ldots + a_n$.

Then, $y_{n-1} = 2.\ n\ (n-1) \ldots 2.x + a_1 (n-1)!$

$= 2.\ n!\ x + a_1 (n-1)!$

Since, we do not know the coeff. Of x^{n-1}. **Answer: (d)**