CIRCLE

THE NATURE OF THE CONIC

Let the equation of the conic be $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\text{where } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

Here different cases arise:

Case 1: $\Delta = 0$ and $ab - h^2 \neq 0 \Rightarrow$ represents a **pair of intersecting lines**. If $a + b = 0$, then the lines are at right angles.

Case 2: $\Delta = 0$ and $ab - h^2 = 0$ i.e. $\frac{a}{h} = \frac{h}{b} = \frac{g}{f} \Rightarrow$ represents a **pair of parallel straight lines**.

Case 3: $\Delta \neq 0$, $a = b$ and $h = 0 \Rightarrow$ represents is a **circle.**

Case 4: $\Delta \neq 0$ and $h^2 = ab$. $\Rightarrow$ represent a **parabola** as the coordinates of the centre are infinite and the terms in second degree make a perfect square.

Case 5: $\Delta \neq 0$ and $h^2 < ab$. $\Rightarrow$ represents an **ellipse**.

Case 6: $\Delta \neq 0$ and $h^2 > ab$. $\Rightarrow$ represents a **hyperbola**.

Case 7: $\Delta \neq 0$ and $h^2 > ab$ and $a + b = 0 \Rightarrow$ represents **rectangular hyperbola.**

DIFFERENT FORMS OF EQUATION OF CIRCLE: -

(i) **CENTRAL or STANDARD FORM**

$$(x - h)^2 + (y - k)^2 = r^2$$

Where centre is (h, k) and radius is 'r'

If centre of the circle is at the origin and radius is 'r', then equation of the circle is

$$x^2 + y^2 = r^2$$

This is also known as the **simplest form.**

(ii) **DIAMETRIC FORM**

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Where (x_1, y_1) & (x_2, y_2) are diametric points.

(iii) **GENERAL FORM**

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Its centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$, where $r > 0$

Note:

- A circle if $g^2 + f^2 - c > 0$
- A point circle $(-g, -f)$ if $g^2 + f^2 - c = 0$
- An empty set if $g^2 + f^2 - c < 0$

(iv) **PARAMETRIC FORM**

$$x = r\cos\theta, \text{ and } y = r\sin\theta$$

are the parametric form of the equation $x^2 + y^2 = r^2$
where θ is the parameter and $0 \le \theta < 2\pi$.
In other words we can say,
Any point on the circle $x^2 + y^2 = r^2$ is $(r\cos\theta, r\sin\theta)$
Parametric form of the equation $(x-h)^2 + (y-k)^2 = r^2$ are

$$x = h + r\cos\theta \text{ and } y = k + r\sin\theta,\ 0 \le \theta \le 2\pi$$

POSITION OF A POINT WITH RESPECT TO A CIRCLE

$S = x^2 + y^2 + 2gx + 2fy + c = 0$
Let (x_1, y_1) be any point
$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
The point lie

- **Outside** the circle if $S_1 > 0$.
- **On** the circle if $S_1 = 0$.
- **Inside** the circle if $S_1 < 0$.

LINE & CIRCLE

If S = 0 be a circle with radius r and L = 0 be a line, & d is the distance of the line L from the centre of the circle S, then,

(a) if $d = r \Rightarrow$ L touches S
(b) if $d > r \Rightarrow$ L does not meet S
(c) if $d < r \Rightarrow$ L meets S in two distinct points and then,

Length of chord intercepted = $2\sqrt{r^2 - d^2}$ **.**

Note: A line L is a normal to a circle S iff it passes through the centre of the circle.

Intersection of Line and Circle

Let line be $y = mx + c$ and the circle be $x^2 + y^2 = a^2$, then,

- if $c^2 < a^2(1 + m^2) \Rightarrow$ line cuts circle in **two real and distinct points.**
- if $c^2 = a^2(1 + m^2) \Rightarrow$ line cuts circle in **two real and coincident points** i.e. line is **tangent** to circle.
- if $c^2 > a^2(1 + m^2) \Rightarrow$ line cuts circle in **two imaginary points** i.e. don't intersect.

Note: The equation of a circle passing through the point of intersection of circle S = 0 and line L = 0 is given by

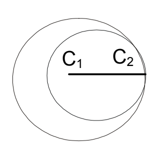
$$S + \lambda L = 0$$

Where λ is any constant.

TWO CIRCLES

Two circles S = 0 and S' = 0 with centers C_1 and C_2 and radii r and r' respectively

(a) **Touch Internally** if $|r - r'| = |C_1C_2|$



In this case, the point of contact divides | C_1C_2 | **Ex**ternally is ratio *r* : *r '*.
Number of tangents = 1

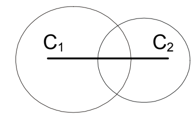
(b) **Intersect in two distinct points** if **| r – r' | < | C_1C_2 | < | r + r' |**

Number of tangents = 2

Note:

- Intersect orthogonally if $r^2 + r'^2 = C_1C_2^2$.
- Intersect at an angle θ given by

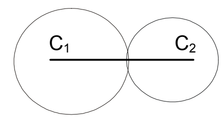
$$\cos\theta = \frac{r^2 + r'^2 - C_1C_2^2}{2rr'}$$



(c) **Touch Externally** if **| r + r' | = | C_1C_2 |**

In this case, the point of contact divides | C_1C_2|, internally in the ratio r : r'.

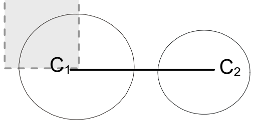
Number of tangents = 3



(d) **Do not intersect** if – 2 cases

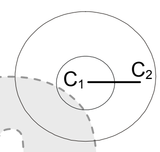
- **| C_1C_2 | > | r + r' |**

  Number of tangents = 4



- **| C_1C_2 | < | r – r' |**

  Number of tangents = 0



CONCENTRIC CIRCLES

Any circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$x^2 + y^2 + 2gx + 2fy + \lambda = 0, \qquad \lambda \in R.$$

i.e. only constant term is different.

ORTHOGONAL CIRCLES

If two circles intersect at right angle i.e., the tangent at their point of intersection are at right angles, then the circles are called ***orthogonal circles***.

The circles

$$S = x^2 + y^2 + 2gx + 2fy + c = 0 \text{ and}$$
$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

Cut orthogonally iff

$$\mathbf{2gg' + 2ff' = c + c'}$$

<u>**Note**</u>**: The equation of a circle passing through the point of intersection** of circles S = 0 and $S_1 = 0$ is given by

$$\mathbf{S + \lambda S_1 = 0}$$

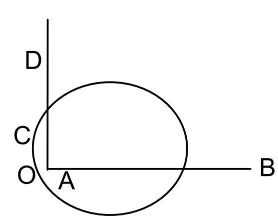
Where λ (≠ –1) is any constant.

<u>**INTERCEPTS ON AXIS**</u>

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$….. (i)

and it cuts the axes at A, B C and D.

Since the circle cut the x-axis at A and B, so put y = 0 in (i),



We get $x^2 + 2gx + c = 0$…….(ii)
Let x_1 and x_2 be the roots of second degree equation (ii)
$\therefore x_1 + x_2 = -2g, \quad x_1 x_2 = c.$
$\therefore$ AB = OB – OA = $x_2 - x_1 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = \sqrt{4g^2 - 4c}$

Intercept on x-axis is AB = $2\sqrt{g^2 - c}$

Similarly putting x = 0 in equation (i), we get intercept on y-axis CD = $2\sqrt{f^2 - c}$

Thus,

$$\textbf{X – INTERCEPT} = 2\sqrt{g^2 - c}$$

$$\textbf{Y – INTERCEPT} = 2\sqrt{f^2 - c}$$

<u>CHORD OF A CIRCLE</u>

The equation of a chord whose mid-point be given

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and (x_1, y_1) be middle point of the chord, then Equation of the chord is

$$S_{11} = S_1$$

i.e. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

<u>Examples:</u>

Ex. Match the List I with List II

List I

Conditions

(a) Centre of circle (4, 5) radius of circle = 7

(b) Circle passes through origin and cuts intercepts 4 and 3 on the axes

(c) Circle touches both the axes and its centre is (4, 4)

(d) Circle touches the y-axis and its centre is (3, 5)

List II

Equation of circle

(A) $x^2 + y^2 - 6x - 10y + 25 = 0$ (B) $x^2 + y^2 - 4x - 3y = 0$

(C) $x^2 + y^2 - 8x - 10y = 8$ (D) $x^2 + y^2 - 8x - 8y + 16 = 0$

The correct match is:

(A) abcd – 1342 (B) abcd – 3421 (C) abcd – 4213 (D) abcd – 3241

Sol. (A) the equation of circle whose centre (4, 5) and radius 7 is $(x-4)^2 + (y-5)^2 = (7)^2$

$\Rightarrow x^2 + y^2 - 8x - 10y = 8.$

(B) Given that OA = 4, and OB = 3

$\therefore$ Centre of the circle is $\left(2, \frac{3}{2}\right)$

And radius OC = $\sqrt{OM^2 + MC^2} = \sqrt{(2)^2 + \left(\frac{3}{2}\right)^2} = \frac{5}{2}$

$\therefore$ The equation of circle whose centre is (2, 3/2) and radius is 5/2 given by

$(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{5}{2}\right)^2 \Rightarrow x^2 + y^2 - 4x - 3y = 0$

(C) Centre of the circle is (4, 4) radius of the circle = AC = BC = 4

$\therefore$ Equation of required circle is

$(x-4)^2 + (y-4)^2 = (4)^2 \Rightarrow x^2 + y^2 - 8x - 8y + 16 = 0$

(D) Centre of the circle is (3, 5) radius of the circle AC = 3

$\therefore$ Equation of the required circle $(x-3)^2 + (y-5)^2 = (3)^2$ $x^2 + y^2 - 6x - 10y + 25 = 0$

$\therefore$ The correct answer is (D)

Ex. If y = 2x be a chord of the circle $x^2 + y^2 = 10x$. Then the equation of the circle whose diameter lies on this line is:

(A) $x^2 + y^2 - 4x - 2y = 4$ (B) $x^2 + y^2 - 4x - 2y = 0$

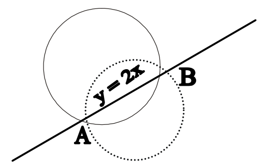
(C) $x^2 + y^2 - 2x - 4y = 0$ (D) $x^2 + y^2 - 4x + 4y = 4$

Sol. The equation of given circle is $x^2 + y^2 = 10x$ … (i)

And its chord is y = 2x … (ii)

Let the chord intersect the given circle at A and B.

Solving (i) and (ii) we get A (0, 0), B (2, 4).



$\therefore$ Equation of the circle joining the line A (0, 0) and B(2, 4) as diameter is

$(x-0)(x-2) + (y-0)(y-4) = 0$ or,

$x^2 + y^2 - 2x - 4y = 0$.

$\therefore$ The correct answer is (C).

Ex. Which of the following is true?

(A) The equation of the circle passing through (1, 1), (2, – 1) and (3, 2) is $x^2 + y^2 - 5x - y + 4 = 0$.

(B) The intercept on x-axis by the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ is 9.

(C) The intercept on y-axis by the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ is 15.

(D) All of the above

Sol. (A) The circle passing through (1, 1), (2, 1) and (3, 2) is given by $x^2 + y^2 - 5x - y + 4 = 0$.

$\therefore$(a) is true.

(B) The equation of circle is $x^2 + y^2 - 5x - 13y - 14 = 0$.

Intercept on x-axis = $2\sqrt{g^2 - c} = 2\sqrt{\left(\frac{-13}{2}\right)^2 - (-14)} = 9$.

Intercept on y-axis = $2\sqrt{f^2 - c} = 2\sqrt{\left(\frac{-5}{2}\right)^2 - (-14)} = 15$.

$\therefore$ (B) and (C) are also true.

$\therefore$ The correct answer of this question is (D).

Ex. Which of the following is false?

(A) The line y = x + 2 touches the circle $x^2 + y^2 = 2$

(B) The equation of the line which is inclined to the x-axis at an angle 30^0 to the circle $x^2 + y^2 = 25$ is $\sqrt{3}y = x + 10$

(C) The equation of tangent which is inclined to the x-axis at an angle 30^0 to the circle $x^2 + y^2 = 25$ is $\sqrt{3}y = x + 10$

(D) At least one of the above is false.

Sol. (A) The centre and radius of the circle $x^2 + y^2 = 2$ are (0, 0) and $\sqrt{2}$ respectively.

The line y = x + 2 touches the circle if the length of perpendicular from (0, 0) on y = x + 2 is equal to the radius i.e. $\frac{|0+0+2|}{\sqrt{1^2+1^2}} = \sqrt{2}$ $\therefore$ (a) is correct.

(B) Given that $\theta = 30^0$ $\therefore m = \tan 30^0 = 1/\sqrt{3}$.

$\therefore$ The line y = m x + c will be a tangent to the circle $x^2 + y^2 = 25$ if $c = \pm a\sqrt{1+m^2}$

or $c = \pm 5\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \pm \frac{10}{\sqrt{3}}$.

$\therefore$ Equation of tangents are $y = \frac{1}{\sqrt{3}}x \pm \frac{10}{\sqrt{3}}$ or $\sqrt{3}y = x + 10$ and $\sqrt{3}y = x - 10$.

$\therefore$ (A), (B) and (C) all are correct. Therefore (D) is false.

$\therefore$ The correct answer of this question is (D).

Ex. Which of the following is false?

(A) The circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 - 2by - c = 0$ intersect orthogonally.

(B) The equation of the circle passing through (2, – 3) and the point of intersection of circles $x^2 + y^2 - 6x + 2y = 5$ and $x^2 + y^2 + 2x + 3y = 7$ is $11x^2 + 11y^2 + 14x + 32y - 75 = 0$.

(C) The equation of the circle passing through the point (– 1, 2) and the point of intersection the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ and the line $3x + 2y = 4$ is $x^2 + y^2 + 2x + 2y - 7 = 0$.

(D) At least one of the above is false.

Sol. (A) The given circles are $x^2 + y^2 + 2ax + c = 0$ … (i)

Here $g_1 = a, f_1 = 0, c_1 = c$

And $x^2 + y^2 - 2by - c = 0$ … (ii)

Here $g_2 = 0, f_2 = -b, c_2 = -c$.

The circles (i) and (ii) cut orthogonally

if $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$\Rightarrow 2(a).\ 0 + 2.\ 0.\ (-b) = c - c \Rightarrow 0 = 0.$ $\therefore$ (A) is true.

(B) The given circles are $x^2 + y^2 - 6x + 2y = 5$ … (i)

and $x^2 + y^2 + 2x + 3y = 7$ … (ii)

$\therefore$ The equation of the circle through the point of intersection of the circles (i) and (ii) is given by

$(x^2 + y^2 - 6x + 2y = 5) + \lambda\,(x^2 + y^2 + 2x + 3y = 7) = 0$ … (iii).

Since it passes through (+2, – 3)

$\therefore (4 + 9 - 12 - 6 - 5) + \lambda(4 + 9 + 4 - 9 - 7) = 0$

or $-10 + \lambda = 0$ or $\lambda = 10$.

or $11x^2 + 11y^2 + 14x + 32y - 75 = 0$.

$\therefore$ (B) is true.

(C) The given circle is $x^2 + y^2 - 4x - 2y + 1 = 0$ and the given line is $3x + 2y = 4$.

$\therefore$ The equation of the circle through the point of intersection of the given circle and the given line is $(x^2 + y^2 - 4x - 2y + 1) + \lambda(3x + 2y - 4) = 0$ … (i)

Since it passes through the point (–1, 2),

$\therefore (1 + 4 + 4 - 4 + 1) + \lambda(-3 + 4 - 4) = 0 \Rightarrow 6 - 3\lambda = 0 \Rightarrow \lambda = 0$

or $x^2 + y^2 + 2x + 2y - 7 = 0$ [from (i)]

$\therefore$ All (A), (B) and (C) are true. $\therefore$ (D) is false.

$\therefore$ The correct answer of the question is (D).

TANGENTS

Condition of tangency: y = mx + c is tangent to $x^2 + y^2 = a^2$, if $c = \pm a\sqrt{1+m^2}$

Point of contact is $\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$

TANGENT AT A POINT

The equation of the tangent to a circle S = 0 at the point (x_1, y_1) is

$$S_{11} = 0$$

Where S_{11} is obtained from S by replacing

$x^2 \rightarrow xx_1$, $y^2 \rightarrow yy_1$, $(x + x_1) \rightarrow \frac{x+x_1}{2}$ and $(y + y_1) \rightarrow \frac{y+y_1}{2}$

So Equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note: The same rule is applicable to write down the equation of tangent to a curve S = 0 (not necessarily a circle) at a given point (x_1, y_1) of the curve. If the equation of the curve contains a term involving xy, then write $\frac{xy_1 + yx_1}{2}$ for xy.

TANGENTS FROM A GIVEN POINT

Algorithm to obtain the equations of tangents drawn from a given point to a given circle.

Step 1: Obtain the point, say (x_1, y_1)

Step 2: Write a line passing through (x_1, y_1) having slope m i.e. $y - y_1 = m(x - x_1)$

Step 3: Equate the length of the perpendicular from the center of the circle to the line in Step II to the radius of the circle.

Step 4: Obtain the value of m from the equation in step III.

Step 5: Substitute m in the equation in step II

Ex. Find the equation of the tangents through (7, 1) to the circle $x^2 + y^2 = 25$.

Sol. The equation of any line through (7, 1) is $y - 1 = m(x - 7) \Rightarrow mx - y - 7m + 1 = 0$

The coordinates of the center and radius of the given circle are (0, 0) and 5 respectively.

The line (i) will touch the given circle if,

Length of the perpendicular from the center = radius

$$\Rightarrow \left|\frac{m \times 0 - 0 - 7m + 1}{\sqrt{m^2 + (-1)^2}}\right| = 5 \quad \Rightarrow \left|\frac{1-7m}{\sqrt{m^2+1}}\right| = 5$$

$$\Rightarrow \frac{(1-7m)^2}{m^2+1} = 25 \quad \Rightarrow 24m^2 - 14m - 24 = 0$$

$$\Rightarrow 12m^2 - 7m - 12 = 0 \quad \Rightarrow (4m + 3)(3m - 4) = 0 \Rightarrow m = -\frac{3}{4}, \frac{4}{3}$$

Substituting the values of m in (i), we obtain

$-\frac{3}{4}x - y + \frac{21}{4} + 1 = 0$ and $\frac{4}{3}x - y - \frac{28}{3} + 1 = 0$

$\Rightarrow$ $3x + 4y - 25 = 0$ and $4x - 3y - 25 = 0$, which are the required equations.

NORMAL TO A CIRCLE AT A GIVEN POINT

Normal The normal at any point on a curve is a straight line which is perpendicular to the tangent to the curve at the point.

Algorithm to find the normal to a circle at a given point (x_1, y_1)

Step 1 Write the equation of the tangent to the given circle at the given point (x_1, y_1).
Step 2 Write the equation of a line perpendicular to the tangent in step 2 and passing through (x_1, y_1).
The equation obtained in step 2 is the required equation of the normal at (x_1, y_1).

Ex. Find the equation of the normal to the circle $3x^2 + 3y^2 - 4x - 6y = 0$ at the point (0, 0).
Sol. The equation of the tangent to the circle $3x^2 + 3y^2 - 4x - 6y = 0$ at (0, 0) is
$3(0 \times x) + 3(0 \text{ x } y) - 2(x + 0) - 3(y + 0) = 0 \quad \Rightarrow 2x + 3y = 0$
Slope of the tangent = –2/3.
So, the slope of the normal at (0, 0) = 3/2.
Hence, the equation of the normal at (0, 0) is
$y - 0 = (3/2)(x - 0)$
$\Rightarrow 3x - 2y = 0$

LENGTH OF TANGENT

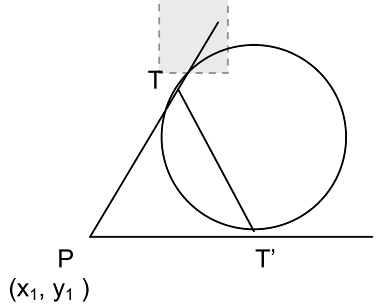
Length of the tangent from point (x_1, y_1) to the circle S is = $\sqrt{S_1}$

PAIR OF TANGENTS

The equation of pair of tangent from a point (x_1, y_1) to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is $\mathbf{SS_1 = S_{11}^2}$
Where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and
$S_{11} = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

CHORD OF CONTACT

If from a point $P(x_1, y_1)$ outside a circle tangents PT and PT' are drawn which touch the circle at T and T' respectively, then the line TT' is called the chord of contact from P.
Equation of chord of contact is $S_{11} = 0$

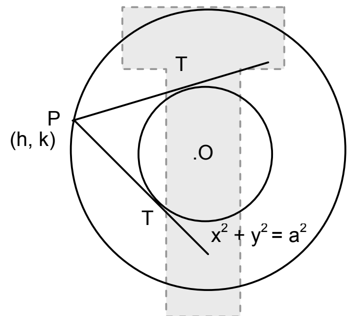


DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents to a given circle is called its **direction circle.**

For the circle $x^2 + y^2 = a^2$, the equation of the director circle is $\mathbf{x^2 + y^2 = 2a^2}$

which is a circle with the same **centre** i.e. **(0, 0)** and **radius = a** $\sqrt{2}$ **.**



COMMON CHORD OF TWO CIRCLE

Definition: The chord joining the points of intersection of two given circles is called their common chord.

Equation of Common Chord

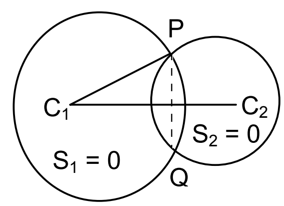
The equation of the common chord of two circles

$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

is $\mathbf{S_1 - S_2 = 0}$

i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$



Remark: If the length of the common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

Examples

Ex. The equation of pair of tangent from the point (1, 2) to the circle $2x^2 + 2y^2 - 8x + 12y + 21 = 0$ is:

(A) $45x^2 - 3y^2 + 2xy - 130x - 8y + 73 = 0$ (B) $45x^2 - 3y^2 + 20xy - 130x - 9y + 71 = 0$

(C) $3x^2 - 45y^2 + 20xy - 130x - 8y + 1 = 0$ (D) none of these

Sol. The equation of the given circle is $2x^2 + 2y^2 - 8x + 12y + 21 = 0$.

The equation of pair of tangent from (1, 2) to the circle is

$$\left(x^2 + y^2 - 4x + 6y + \frac{21}{2}\right)\left(1^2 + 2^2 - 4.1 + 6.2 + \frac{21}{2}\right) = \left[x.1 + y.2 - 2(x+1) + 3(y+2) + \frac{21}{2}\right]^2$$

or $45x^2 - 3y^2 + 2xy - 130x - 8y + 73 = 0$

$\therefore$ The correct answer is (A).

Ex. The equation of the circle is $x^2 + y^2 - 2x - 4y + 3 = 0$, then pick the false statement.

(A) tangent at (2, 3) is $x + y = 5 = 0$ (B) normal at (2, 3) is $x - y + 1 = 0$

(C) length of tangent from the point (4, 5) is 4 (D) tangent at (1, 3) is $x + y - 4 = 0$

Sol. The given equation of circle is $x^2 + y^2 - 2x - 4y + 3 = 0$ … (1)

$\therefore$ Tangent at (2, 3) is $x.2 + y.3 - 1(x + 2) - 2(y + 3) + 3 = 0$ or $x + y - 5 = 0$

∴ (A) is correct.

Now slope of tangent at (2, 3) = –1.

∴Slope of normal at (2, 3) = –1/ –1 = 1.

∴Equation of normal at (2, 3) is (y – 3) = 1 (x – 2) or x – y + 1 = 0.

(B) is correct.

Now the length of tangent on circle (1) from (4, 5) = $\sqrt{4^2+5^2-2(4)-2(5)+3}$ = 4

∴ (C) is correct.

Now the tangent to the circle (1) at (1, 3) is x.1 + y. 3 – (x + 1) –2(y + 3) + 3 = 0 or, y = 4

∴ (D) is not correct.

∴ Only (A), (B) and (C) are correct

∴The correct answer is (D).

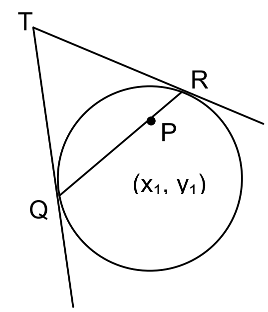
POLE AND POLAR

From a point P draw two chords to the circle, then the locus of the point of intersection of tangents drawn at the **Ex**tremities of the chords is called the **polar** of P. The point P is called the **pole** of its polar.

The Polar of point P(x_1, y_1) with respect to circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

S_{11} = 0

(**Note:** Same as equation of tangent)



Conjugate Points:

Two points are called conjugate points with respect to a circle if each point lies on the polar of the other point with respect to the same circle.

Conjugate lines:

Two lines are called conjugate lines with respect to a circle if the pole of each line lies on the other line.

RADICAL AXIS

The radical axis of two circles is the locus of a point from which the length of tangents drawn to the circles is equal.

Equation of Radical Axis:

Let the equations of circle be $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

Let P(x_1, y_1) be any point on the radical axis so that the length of tangents from P to the circle be equal.

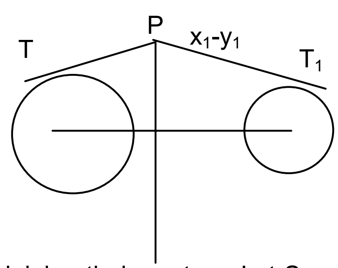
i.e., $PT^2_1 = PT^2_2$ or, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$= x_1^2 + y_1^2 + 2g_1 x_1 + 2f_1 y_1 + c_1$ or,

$2(g_1 – g_2) x + 2(f_1 – f_2) y + c_1 – c_2 = 0$

or **$S_1 – S_2 = 0$**

This is the equation of radical axis which is a straight line.



Result about radical axis :

The radical axis of two circles is always perpendicular to the line joining their centres. Let $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ be three circles. Then the radical axes taken in pairs are either parallel or concurrent.

Radical axis are parallel if the centres of the circles are collinear.

Radical Centre :

The point of concurrence of the radical axes of three circles, whose centres are non-collinear taken in pairs, is called the radical centre of the circles. The circle with centre at the radical centre and radius equal to the length of the tangent from it to any one of the circles intersects all the three circles orthogonally.

COAXIAL SYSTEM OF CIRCLES

A system of circles, every pair of which has the same radical axis is called a coaxial system of circles. Circles passing through two fixed points form a coaxial system of circles, because every pair of circles has the same common chord and therefore the same radical axis.

$\therefore$**The centers of circles of a coaxial system must lie on a line perpendicular to the common radical axes.**

Limiting points

The point circles of a coaxial system are called the limiting points. Thus the centre of the circle of coaxial system when its radius is equal to zero is the limiting point of the system.

Equation of co axial system when

(i) the equation of the radical axis and of one circle of the system are given:

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle and $P \equiv lx + my + n = 0$ be the radical axis. Then $S + kP = 0$

(k is an arbitrary constant) represents the coaxial system of circles

(ii) when the equation of any two circles of the system are given:

Let $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ be two circles of the coaxial system. Then $S_1 + kS_2 = 0$ $(k \neq -1)$ represents the coaxial system.

Also $S_1 + k(S_1 - S_2) = 0$

Equation of a system of coaxial circles in the simplest form:

$x^2 + y^2 + 2gx + c = 0$, where g is a variable and c a constant. The common radical axis is the y-axis

Orthogonal System

The orthogonal system of a coaxial system is a system of coaxial circles whose every circle cut every circle of the given coaxial system orthogonally.

Ex. The radical centre of the circles $x^2 + y^2 + 2y + 3 = 0$, $x^2 + y^2 + 2x + 4y + 5 = 0$ & $x^2 + y^2 + kx - 8y - 9 = 0$ is $\left(-\frac{2}{3}, \frac{2}{3}\right)$. What is the value of k?

Sol. The given circles are $x^2 + y^2 + 2y + 3 = 0$ … (i)

$x^2 + y^2 + 2x + 4y + 5 = 0$ … (ii)

and $x^2 + y^2 + kx - 8y - 9 = 0$ … (iii)

The radical axis of (i) and (ii) is

$(x^2 + y^2 + x + 2y + 3) - (x^2 + y^2 + 2x + 4y + 5 = 0)$ or $x + 2y + 2 = 0$ … (iv)

The radical axis of (ii) and (iii) is $(2 - k)\,x + 12 + 14 = 0$ … (v)

Solving (iv) and (v), we get $x = \frac{2}{k+4}$ and $y = \frac{k+5}{k+4}$.

$\therefore$ The radical centre of the given circles is $\left(\frac{2}{k+4}, \frac{k+5}{k+4}\right)$.

But given radical centre is (–2/3, (–2/3) $\therefore$ $\frac{2}{k+4} = -\frac{2}{3} \Rightarrow k = -7$.

IMPORTANT FORMULA OF CIRCLE	
1. Central form with center (h, k) & radius = r	$(x – h)^2 + (y – k)^2 = r^2$.
2. Simplest form or the standard form center as origin & radius =r	$x^2 + y^2 = r^2$
3. Diametric points (x_1, y_1) & (x_2, y_2) are given then	Equation is $(x–x_1)(x–x_2) + (y–y_1)(y–y_2) = 0$
4. Length of chord intercepted (r = radius, d = distance of origin from chord)	$2\sqrt{r^2 - d^2}$
5. Intercept on x-axis, y-axis	$2\sqrt{g^2 - c}$, $2\sqrt{f^2 - c}$.
6. Any circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\lambda \in R$.	$x^2 + y^2 + 2gx + 2fy + \lambda = 0$ Where $\lambda \in R$.
7. Equation of tangent at given point (x_1, y_1) (**Note** this also =n of polar & chord of contact)	$S_{11} = 0$ or $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
8. Equation of circle from a given point (x_1, y_1) lies outside the circle.	**Steps 1:** Write = n of tangent as $y = mx + c$ or $y – y_1 = m(x – x_1)$ **Steps 2:** Equate the length of tangent to line to the radius of circle & obtain the value of m.
9. The equation of pair of tangent from a point (x_1, y_1).	$SS_1 = T^2 = (S_{11})^2$ where $S = x^2 + y^2 + 2gx + 2fy + c$, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $S_{11} = T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
10. Length of tangent from (x_1, y_1)	$\sqrt{S_1}$
11. Chord of contact from (x_1, y_1)	$S_{11} = 0$ (same as =n of tangent at a given pt)
13. Position of point (x_1, y_1)	if $S_1 = 0$ point lies on the circle if $S_1 > 0$ outside circle & if $S_1 < 0$ inside
14. Line $y = mx + c$ And Circle $x^2 + y^2 = r^2$	Cuts at two pts if $c^2 < a^2(1 + m^2)$ & Is a tangent if $c^2 = a^2(1 + m^2)$ Never cut if $c^2 > a^2(1 + m^2)$
15. Angle of Intersection of two circles (d = distance b/w centers)	$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}$
16. Orthogonal circles $x^2 + y^2 + 2gx + 2f + c = 0$ and $x^2 + y^2 + 2g_1x + 2f_1 + c_1 = 0$ are orthogonal if	$r_1^2 + r_2^2 = d^2$ Or $2gg_1 + 2ff_1 = c + c_1$
17. The equation of a circle passing through the point of intersection of circles S = 0 and $S_1 = 0$ is given by	$S + \lambda S_1 = 0$ where λ ($\neq -1$) is any constant.
18. The equation of a circle passing through the point of intersection of circle . S = 0 and line L = 0 is given by	$S + \lambda L = 0$ where λ, is any constant
19. Equation of radical axis or common chord of circle $S_1 = 0$ & $S_2 = 0$	$S_1 – S_2 = 0$
20. The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ having (x_1, y_1) as its middle point is	$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$. OR $S_{11} = S_1$

Examples:

Ex. Find the equation of the circle which touches each axis at a distance 5 from the origin.

Sol. As the circle touches both the axis at a distance of 5, its centre will be (5, 5) and radius will also be 5.
So the required equation is $(x - 5)^2 + (y - 5)^2 = (5)^2$.
Or $x^2 + y^2 - 10x - 10y + 25 = 0$.

Ex. Find the equation of the circle which touches both axes and passes through the point (–2, –3).

Sol. Let the radius of the circle be a. As it touches both the axes, the centre may be ($\pm$a, $\pm$ a).
Again the circle passes through (– 2, – 3), so it lies in the 3rd quadrant, hence the centre will be (– a, – a), and its distance from (– 2, – 3) will be equal to radius a.
So $(-a + 2)^2 + (-a + 3)^2 = a^2$
or $a^2 + 4 - 4a + a^2 - 6a + 9 = a^2$
or $a^2 - 10a + 13 = 0$
or $a = \dfrac{10 \pm \sqrt{(100 - 52)}}{2}$ or $a = 5 \pm \sqrt{2}$.
Therefore the required equation is
$\left\{x + \left(5 \pm \sqrt{2}\right)\right\}^2 + \left\{y + \left(5 \pm \sqrt{2}\right)\right\}^2 = \left\{5 \pm \sqrt{2}\right\}^2$
or $x^2 + y^2 + 2(5 \pm \sqrt{2})(x + y) = 37 \pm 10\sqrt{2}$

Ex. Find the equation of the circle concentric with the circle $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ and having its area equal to 16π square units.

Sol. The equation of the given circle is $2x^2 + 2y^2 + 8x + 10y - 39 = 0$
or $x^2 + y^2 + 4x + 5y - 39/2 = 0$.
The coordinates of its centre are (– 2, – 5/2).
Let the area of the required circle be πr^2, where r is its radius.
So, Area = 16π
$\Rightarrow \pi r^2 = 16 \Rightarrow r = 4$.
Hence, the equation of the required circle is $(x + 2)^2 + (y + 5/2)^2 = 4^2$

Ex. Show that the points (9, 1), (7, – 9), (– 2, 12) and (6, 10) are concyclic.

Sol. Let the equation of the circle passing through (9, 1), (7, 9) and (– 2, 12) be
$x^2 + y^2 + 2gx + 2fy + c = 0$ … (i)
Then $82 + 18g + 2f + c = 0$ … (ii)
$130 + 14g + 18f + c = 0$ ... (iii)
$148 - 4g + 24f + c = 0$ … (iv).
Subtracting (ii) from (iii), we get $48 - 4g + 16f = 0 \Rightarrow 12 - g + 4f = 0$ ….. (v)
Subtracting (iii) from (iv), we get $18 - 18g + 6f = 0 \Rightarrow 3 - 3g + f = 0$ …. (vi)
Solving (v) and (vi) as simultaneous linear equation in g and f, we get f = – 3, g = 0.
Putting f = – 3, g = 0 in (ii), we get 82 + 0 – 6 + c = 0 = – 76.
Substituting the values of g, f, c in (i), we get $x^2 + y^2 - 6y - 76 = 0$ as the equation of the circle passing through (9, 1), (7, 9) and (– 2, 12).
Clearly point (6, 10) satisfies this equation. Hence the given points are concyclic.

Ex. Find the equation of the circle drawn on the intercept made by the line 2x + 3y = 6 between the coordinate axes as diameter.

Sol. The line 2x + 3y = 6 meets X and Y axes at A (3, 0) and B (0, 2) respectively.
Taking AB as a diameter, the equation of the required circle is
$(x – 3) (x – 0) + (y – 0) (y – 2) = 0$
[Using $(x – x_1) (x – x_2) + (y – y_1) (y – y_2) = 0$]
$\Rightarrow x^2 + y^2 – 3x – 2y = 0$.

Ex. For what value of k will the straight line 3x + 4y = k touch the circle $x^2 + y^2 = 10x$?

Sol. The equation of the given line is $3x + 4y – k = 0$ … (i)
The equation of the given circle is $x^2 + y^2 = 10x$ or $x^2 – 10x = 0$ … (ii)
The coordinates of the centre and radius of (ii) are (5, 0) and 5 respectively
If (i) touches the circle (ii), then (length of the ⊥ from (5, 0) to 3x + 4y – k = 0) = radius

$$\Rightarrow \left|\frac{3\times 5+4\times 0-k}{\sqrt{3^2+4^2}}\right| = 5 \Rightarrow \left|\frac{15-k}{5}\right| = 5$$

$\Rightarrow 15 – k = \pm 25 \Rightarrow k = –10$ or $k = 40$.

Ex. Find the equations of tangents through (7, 1) to the circle $x^2 + y^2 = 25$.

Sol. The equation of any line through (7, 1) to $y – 1 = m (x – 7) \Rightarrow mx – y – 7m + 1 = 0$ … (i)
The coordinates of the centre and radius of the given circle are (0, 0) and 5 respectively
The line (i) will touch the given circle if,
Length of the perpendicular from the centre = radius

$$\Rightarrow \left|\frac{m\times 0-0-7m+1}{\sqrt{m^2+(-1)^2}}\right| = 5 \Rightarrow \left|\frac{1-7m}{\sqrt{m^2+1}}\right| = 5 \Rightarrow \frac{(1-7m)^2}{m^2+1} = 25 \Rightarrow 24m^2 - 14m - 24 = 0$$

$\Rightarrow 12m^2 – 7m – 120 = 0 \Rightarrow (4m + 3) (3m – 4) = 0$
$\Rightarrow m = – 3/4, 4/3$.
Substituting the values of m in (i), we obtain
$3x + 4y – 25 = 0$ and $4x – 3y – 25 = 0$ which are the required equation

Ex. Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 – 5x + 3y – 2 = 0$; find the point of intersection of these tangents.

Sol. The circles are given as $x^2 + y^2 = 12$ … (1)
and $x^2 – 5x + 3y – 2 = 0$ ... (2)
If A and B are the points of intersection of (1) and (2),
Clearly AB will be the common chord whose equation will be $(x^2 + y^2 – 12) – (x^2 + y^2 – 5x + 3y – 2) = 0$
or $5x – 3y – 10 = 0$ … (3)
If P be the point where the tangents at A and B with respect to (1), meet each other, AB will be the chord of contact of P.
Let the co-ordinates of P be (α, β)
Equation of the chord of contact of (α, β) with respect to (1) is $x\alpha + y\beta – 12 = 0$ …… (4)
As (3) and (4) represent the same equation, comparing the coefficients, we get

$\frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10}$, by which, we get $\alpha = 6$ and $\beta = –18/5$.

Hence the required point is (6, –18/5)

Ex. Find the equation of the circle through the intersection of the circles $x^2 + y^2 - 8x - 2y + 7 = 0$ and $x^2 + y^2 - 4x + 10y + 8 = 0$, having its centre on y-axis.

Sol. The required equation will $S_1 + kS_2 = 0$.

Now the x coordinates of the centre must be zero.

This gives $k = -2$.

Hence, the required equation will be $x^2 + y^2 + 22y + 9 = 0$

Ex. Find the equations to the straight lines joining the origin to the points of intersection of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$.

Sol. The equations are given as $x^2 + y^2 - 4x - 2y = 4$ … (1)

and $x^2 + y^2 - 2x - 4y - 4 = 0$ …. (2)

The required equation will be obtained by making (1) homogeneous with the help of the common chord of (1) and (2).

From (1) and (2), subtracting, we get $(x^2 + y^2 - 4x - 2y = 4) - (x^2 + y^2 - 2x - 4y - 4) = 0$ or

$2x - 2y = 0$ or $y - x = 0$.

Hence the equation of the common chord is $y - x = 0$.

As it passes through the origin, the equation of the lines joining points of intersection of (1) and (2) with origin is $(y - x)(y - x) = 0$ or $(y - x)^2 = 0$.